

Set Theory 集合理论

Definition A set is any collection of objects.

Singleton 单元素集: Only one element.

$\emptyset$ 与 $\{\emptyset\}$ 不同, $\emptyset$ 是空集, $\{\emptyset\}$ 是单元素集, 都是集合.

Definition If any element of a set A is also an element of a set B , then we say A is a subset of B .

$A \subset B$ means: $\forall x \in A \Rightarrow x \in B$

唯一的一个证明两个集合相等的方法: $A \subset B, B \subset A \Rightarrow A = B$

" $A = B$ " means: $\forall x \in A \Rightarrow x \in B$ and $\forall x \in B \Rightarrow x \in A$

Proper subset 真子集: A is a proper subset of B means

$\Rightarrow \forall x \in A \Rightarrow x \in B$ and $\exists y \in B$ such that $y \notin A$

Definition The elements of power set $P(X)$ are the subsets of X .

Sometimes the power set is written 2^X .

$P(X)$ is never empty.

$\emptyset \in P(X), X \in P(X). \quad x \in X \Leftrightarrow \{x\} \in P(X). \quad Y \subset X \Leftrightarrow Y \in P(X)$

eg: $P(\emptyset) = \{\emptyset\}, P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$

Definition Union: $A \cup B = \{x: x \in A \text{ or } x \in B\}$

Laws: ① $A \cup B = B \cup A$ ② $(A \cup B) \cup C = A \cup (B \cup C)$ ③ $A \cup A = A$ ④ $A \cup \emptyset = A$ ⑤ $\emptyset \subset A$

Definition Intersection: $A \cap B = \{x: x \in A \text{ and } x \in B\}$

Laws: ① $A \cap B = B \cap A$ ② $(A \cap B) \cap C = A \cap (B \cap C)$ ③ $A \cap A = A$ ④ $A \cap \emptyset = \emptyset$ ⑤ $\emptyset \subset A$

Distribution Laws: ① $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

② $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

③ $A \cup \left(\bigcap_{i \in I} B_i \right) = \bigcap_{i \in I} (A \cup B_i)$

④ $A \cap \left(\bigcup_{i \in I} B_i \right) = \bigcup_{i \in I} (A \cap B_i)$

proof: ④ $\Rightarrow \forall x \in A \cap \left(\bigcup_{i \in I} B_i \right) \Rightarrow x \in A, x \in \bigcup_{i \in I} B_i \Rightarrow \exists i_0, \text{ such that } x \in B_{i_0}$

$\Rightarrow x \in A \cap B_{i_0} \Rightarrow x \in \bigcup_{i \in I} (A \cap B_{i_0})$

Definition Difference ($\setminus$ or $-$): $A \setminus B = \{x: x \in A \text{ and } x \notin B\}$

Definition The difference of the universal set Ω and A is called the complement of A and denoted by A^c , i.e.

$$A^c = \{x: x \notin A\} \quad A^c = \bar{A}$$

(Note: 万有集在谈到补集前要被确定)

$$A^c = \Omega \setminus A = \{x: x \in \Omega \text{ and } x \notin A\} = \{x: x \notin A\} \text{ since } x \in \Omega \text{ is always true.}$$

$$\text{Laws: } ① (A^c)^c = \{x: x \notin A^c\} = \{x: x \in A\}$$

$$② \phi^c = \{x: x \notin \phi\} = \Omega$$

$$③ \Omega^c = \{x: x \notin \Omega\} = \phi$$

De Morgan's Laws: (Important!!!)

$$(A \cup B)^c = A^c \cap B^c \quad \text{并的补等于补的交}$$

$$(A \cap B)^c = A^c \cup B^c \quad \text{交的补等于补的并}$$

↓ infinity

$$\Delta \text{ proof: } \forall x \in (\bigcap_{k=1}^{\infty} A_k)^c$$

$$(\bigcap_{k=1}^{\infty} A_k)^c = \bigcup_{k=1}^{\infty} A_k^c$$

$$\Rightarrow x \notin \bigcap_{k=1}^{\infty} A_k$$

$$(\bigcup_{k=1}^{\infty} A_k)^c = \bigcap_{k=1}^{\infty} A_k^c$$

$$\Rightarrow \exists k_0, x \in A_{k_0}^c$$

↓ generally

$$\Rightarrow x \in \bigcup_{k=1}^{\infty} A_k^c$$

$$(\bigcap_{k=1}^n A_k)^c = \bigcup_{k=1}^n A_k^c$$

$$\forall x \in \bigcup_{k=1}^n A_k^c$$

$$(\bigcup_{k=1}^n A_k)^c = \bigcap_{k=1}^n A_k^c$$

$$\Rightarrow \exists k_0, \text{ s.t. } x \in A_{k_0}^c \Rightarrow x \notin A_{k_0} \Rightarrow x \in (\bigcap_{k=1}^n A_k)^c$$

Definition A function or map f from X to Y is a rule which, for each element of X , specifies exactly one element of Y . We write

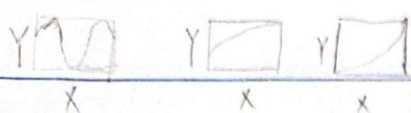
$$f: X \rightarrow Y \text{ or } X \rightarrow Y, x \mapsto f(x) \text{ or } f: X \rightarrow Y, x \mapsto f(x)$$

The set X is called the domain of f and is denoted $\text{dom}(f)$, and Y is the codomain of f . Finally

$$\text{im}(f) := \{y \in Y; (\exists x \in X, \text{ s.t. } y = f(x))\}$$

is called the image of f .

$\text{im}(f)$ 可能只是 Y 的一部分
可以通过 f 这个路径来到



$X \rightarrow Y$ bijective $\Leftrightarrow Y \rightarrow X$ bijective

No.

Date

f is:

card 是元素数, 一个 f 是双射 card 就相同? 对

surjective 满射的: If $\text{im}(f) = Y$; (onto)

injective 单射的: If $f(x) = f(y)$ implies $x = y$ for all $x, y \in X$; (one-to-one)

bijective 双射的: If f is both surjective and injective; (one-to-one correspondence)

Definition Let $f: X \rightarrow Y$ and $g: Y \rightarrow V$ be two maps. Then we define a new map $g \circ f$, the composition of f and g , by

$(g \circ f)(x) = g(f(x))$

Important:

(a) The map $I_X: X \rightarrow X, x \mapsto x$ is the identity map (of X). I_X sometimes written for I .

(b) If $X \subset Y$, then $i: X \rightarrow Y, x \mapsto x$ is called the inclusion of X into Y . 在 $X \rightarrow P(X)$ 中用到了

(c) If $f: X \rightarrow Y$ and $A \subset X$, then $f|_A: A \rightarrow Y, x \mapsto f(x)$ is the restriction of f for A . Clearly $f|_A = f \Leftrightarrow A = X$. 限制

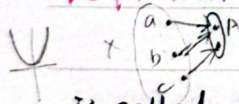
这里的等号不是简单地代表 image 相同, 而是从哪来, 到哪去, 都一样

Definition Let $f: X \rightarrow Y$ be bijective. Then the inverse function

f^{-1} of f is the unique function $f^{-1}: Y \rightarrow X$ such that $f \circ f^{-1} = I_Y$ and $f^{-1} \circ f = I_X$

prop: Let $f: X \rightarrow Y$ and $g: Y \rightarrow V$ be bijective. Then $g \circ f: X \rightarrow V$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. 双向调换, 作用

Definition Let $f: X \rightarrow Y$ be a function and



$$f^{-1}(C) := \{x \in X; f(x) \in C\}$$

is called the pre-image of C and

Note: Pre-image always exists, not

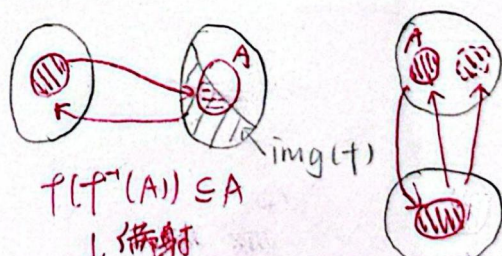
虽然和逆映射的符号相同, 但表示不同

For a map $f: X \rightarrow X, f(f^{-1}(A)) \subseteq A$

eg: $f: \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$

$$\textcircled{1} A = \mathbb{R}, f^{-1}(A) = [0, \infty), f(f^{-1}(A)) = [0, \infty) \subset A$$

由此可以看出, 不用逆映射, $f(x)$ 只需要被 A 包含



$$f(f^{-1}(A)) \subseteq A$$

$\downarrow$ 满射

$$f(f^{-1}(A)) = A$$

$$f^{-1}(f(A)) \supseteq A$$

$\downarrow$ 单射

$$f^{-1}(f(A)) = A$$

$\uparrow$ 映射

KOKUYO

双射基数相同的例子

$$X = \{1, 2, 3, \dots\}, Y = \{2, 4, 6, \dots\}$$

$f: X \rightarrow Y, x \mapsto 2x$, f 是一个双射

故尽管 $Y \subset X$, $\text{Card}(X) = \text{Card}(Y)$

Suppose A and B are two sets. Then
and B is:

$$\{A, b \in B\}.$$

Usually, $A \times B \neq B \times A$.

eg. $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a, b) : a \in \mathbb{R} \text{ and } b \in \mathbb{R}\}$

Definition Two sets A and B are said to have the same cardinal number (or equipotent) (if and only if) there exists a bijection between A and B . ↓ 基数

The cardinal number of A is denoted by $\text{Card}(A)$ or $|A|$.

We shall say $\text{Card}(A) \leq \text{Card}(B)$ iff there exists an injection between A and B . 只要存在一个单射即可
有取等 $\iff$

Theorem (Cantor - Schröder - Bernstein) If $\text{Card}(X) \leq \text{Card}(Y)$ and $\text{Card}(Y) \leq \text{Card}(X)$, then $\text{Card}(X) = \text{Card}(Y)$

Definition A set B is called countable if the $\text{Card}(B)$ is the same as the cardinal number of the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$.

In other words, B is countable iff there exists a one-to-one correspondence between the elements of B and $\mathbb{N} = \{1, 2, 3, \dots\}$. 存在可数的无限集

The cardinal number of a countable set is denoted by $\aleph_0$.

Theorem (1) A set B is countable iff all the elements of B can be written as a sequence, i.e. $B = \{x_1, x_2, x_3, \dots\}$. ($\exists$ one-to-one correspondence between B and $\mathbb{N}$)

Theorem (2) If both A and B are countable, then so is $A \cup B$. Moreover, if $A_1, A_2, \dots, A_n$ are all countable, then so is $\bigcup_{i=1}^n A_i$. Even more, if $A_1, A_2, A_3, \dots$ are all countable, then so is $\bigcup_{k=1}^{\infty} A_k$. (用对角线法写成新的序列, 抓住了两边有界, 两边无限但可列的特点) 并集可数

Theorem (3. the "smallest infinite set") Any infinite set contains a countable subset (不断从无限集 A 中提出 $a_1, a_2, a_3, \dots$, 因为无限, $A \setminus \{a_1, a_2, \dots, a_n\} \neq \emptyset$, 可以

一直提) $\xrightarrow{\text{可推出}} \text{Card}(A) \geq \aleph_0$. i.e. $\aleph_0$ 是无限集中的最小基数

Theorem (4) If $A_1, A_2, \dots, A_n$ are all countable sets, then so is the Cartesian product $A_1 \times A_2 \times \dots \times A_n$. In particular, if A is countable, then so is A^n . (和课程2类似) **不能推广至无限**

(proof: 只要证明若 A, B count, $A \times B$ count. 即可)

$$A \times B = \{(a, b); a \in A, b \in B\}$$

对于确定的 a_i , $C_i = \{(a_i, b); b \in B\} \approx B$ 可以双射到 B

故 C_i 也为 count.

而 $A \times B = \bigcup_{i=1}^{\infty} C_i$, 根据 Theorem (2), $A \times B$ count.

注: 当 $A_1, A_2, A_3, \dots$ 都 count., 不能说明 $A_1 \times A_2 \times A_3 \times \dots$ 是 count. 的(?))

Theorem (5) The set of all rational number is countable. (有理数都可以写成分数的形式, 按照每行分母相同, 逐列按自然数列排开)

Theorem (6) The set of all algebraic numbers is countable.

代数数: A real number that is the root of some polynomial with integer coefficients. (有理数一定是代数数: $r = \frac{m}{n}$ 是 $nx - m = 0$ 的根; 有的无理数是代数数: $x^2 - 2 = 0$ 的根是 $\sqrt{2}$)

proof: ① 多项式的根是有限的

② 横竖可分为次数和项的系数进行排序

↓ 可列序列的并集可列

代数数集可列

Definition The cardinal number of the set $[0, 1]$ is denoted by c . ^{infinite and thus $c \geq \aleph_0$.}

Theorem (7) The set $[0, 1]$ is not countable. Hence $c \neq \aleph_0 \Rightarrow c > \aleph_0$.

($[0, 1]$ 中的所有数可以以 $x_1 = 0.a_{11}a_{12}a_{13}\dots a_{1n}$, $x_2 = 0.a_{21}a_{22}a_{23}\dots a_{2n}\dots$, 但可以选取 $a_{x1} \neq a_{11}$, $a_{x2} \neq a_{22}$, $a_{x3} \neq a_{33}\dots$ 使能 x_x 不同于任何序列中的数)

③ 区间套定理 Nested Interval Theorem

Let $I_1 = [a_1, b_1] \supset I_2 = [a_2, b_2] \supset I_3 = [a_3, b_3] \supset \dots$ With $n \rightarrow \infty$, $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is one point.

ex. $[1, \infty) \supset [2, \infty) \supset \dots \supset [n, \infty) \supset \dots$ $\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset$

假设 $[0, 1]$ 可列, $[0, 1] = \{a_1, a_2, a_3, \dots\}$

将 $[0, 1]$ 等分为三个等分 $\left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{2}{3}, 1\right]$

Assm. $a_1 \notin I_1, a_2 \notin I_2 \subset I_1, \dots, a_n \notin I_n \subset I_{n-1}$ ($I_n = \frac{1}{3} I_{n-1}$)

Let $\bigcap_{n=1}^{\infty} I_n = \{a\}$

a 不在序列中 $\Rightarrow$

证毕

可以把 $\text{Card} = C$ 的东西都看成区间

(?) Properties: ① 所有与 A ($\text{Card}(A) = C$) 有关的运算 (和、乘) 都为 C

这些 intervals ② The following sets: $[0, 1], (0, 1), [0, 1), (a, b), \mathbb{R} = (-\infty, +\infty),$
都不可数 $\mathbb{R}_+ = [0, \infty); \mathbb{R}^2; \mathbb{R}^3; \mathbb{R}^n$ have the cardinal number C .

③ $\text{Card}(\{\text{irrational numbers}\}) = C$

proof: 即证 $\text{Card}(\mathbb{R}) = \text{Card}(\mathbb{R} \setminus \mathbb{Q}) = C$ $\mathbb{Q}$ 为有理数 set

即证 $\exists f: \mathbb{R} \rightarrow \mathbb{R} \setminus \mathbb{Q}$ 是一个双射

假设 $f(x) = \begin{cases} x & \text{if } x \in \mathbb{R} \setminus (\mathbb{Q} \cup (\mathbb{Q} + \sqrt{2})) \\ g(x) & \text{if } x \in (\mathbb{Q} \cup (\mathbb{Q} + \sqrt{2})) \end{cases}$

$x \mapsto x$ 是一个双射; 令 $g(x): \mathbb{Q} \cup (\mathbb{Q} + \sqrt{2}) \rightarrow \mathbb{Q} + \sqrt{2}$

因为 $g(x)$ 的 domain 和 codomain 都可以通过与 $\mathbb{N}$ 数列映射联系起来, 故 $g(x)$ 可以是一个双射.

故 f 是一个双射

④ 若 X 不可数, 从 X 中提出一个可数的子集 Q , $X \setminus Q$ 与 X 含有相同的基数 从不可数中提可数, 基数不变

⑤ $\text{Card}(\{\text{transcendental numbers}\}) = C$

思考: 当 X 为无限集时, $\text{Card}(X) \geq \aleph_0$, 故无限集存在最小基数

又因为 $\text{Card}(P(X)) > \text{Card}(X)$, 故无限集不存在最大基数

proof: $\forall x \in X, \{x\} \in P(X)$

$\exists f: X \rightarrow P(X), x \mapsto \{x\}$. f 为单射

$\Rightarrow \text{Card}(P(X)) \geq \text{Card}(X)$

再证 $\text{Card}(P(X)) \neq \text{Card}(X)$ 即可

即证任意映射都不是满射即可

① $X \neq \emptyset \Rightarrow \forall f: X \rightarrow P(X)$ 不是满射, 即存在 $C \in P(X)$ s.t. $\forall x \in X, f(x) \neq C$

Let $C = \{x \in X : x \notin f(x)\}$

② $X = \emptyset, P(X) = \{\emptyset\}$

前提 $\begin{cases} \text{i) if } x \in f(x), \text{ then } x \notin C \Rightarrow C \neq f(x) \\ \text{ii) if } x \notin f(x), \text{ then } x \in C \Rightarrow C \neq f(x) \end{cases}$

$\text{Card}(P(X)) = 1 > \text{Card} \emptyset = 0$

该集合的每一个元素都是一个无穷数列

Theorem: Let $E = \{(a_1, a_2, \dots) : a_i = 0 \text{ or } 1\}$. Then E is uncountable.

Proof: 假设 E 可数, 那么我们可以将其元素进行排列

$$x_1 = (x_1^1, x_1^2, \dots), x_2 = (x_2^1, x_2^2, \dots), \dots$$

Define an element $y = (y_1, y_2, \dots)$ by

$$y_i = \begin{cases} 0, & x_i^i = 1 \\ 1, & x_i^i = 0 \end{cases} \quad \text{其实和实数不可列的证明方法一样}$$

Then $y \in E$ but $y \neq x_i$ for any $i \in \mathbb{N}$. This is a contradiction.

Metric Spaces 度量空间

距离 (p维): $d_p(x, y) = (\sum_{i=1}^p |x_i - y_i|^p)^{\frac{1}{p}}$

Convex 凸的: ① $f: [a, b] \rightarrow \mathbb{R}$ is convex if $f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$

② A set $E \subset \mathbb{R}^k$ is convex if $\lambda x + (1-\lambda)y \in E$ whenever $x \in E, y \in E$, and $0 < \lambda < 1$

③ If f is twice differentiable and $f'' \geq 0$, then f is convex.

举个例子. 球是凸的. For $\|y-x\| < r, \|z-x\| < r$, and $0 < \lambda < 1$,

$$\|\lambda y + (1-\lambda)z - x\| = \|\lambda(y-x) + (1-\lambda)(z-x)\|$$

$$\leq \lambda \|y-x\| + (1-\lambda) \|z-x\|$$

$$< \lambda r + (1-\lambda)r = r$$

Definition A metric space is a pair (X, d) , where X is a set and $d: X \times X \rightarrow [0, \infty)$ for all $x, y, z \in X$ has the following properties:

1. (Positivity) $d(x, y) = 0 \Leftrightarrow x = y$.
2. (Symmetry) $d(x, y) = d(y, x)$
3. (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$

A function $d: X \times X \rightarrow [0, \infty)$ that satisfies these axioms is called a distance function on X .

Discrete metric 离散度量: $d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$, also known as the trivial metric, ^{平凡度量} the space (X, d) is called the discrete metric space.

Minkowski's inequality ^{闵可夫斯基}: For arbitrary complex numbers $x_1, \dots, x_n, y_1, \dots, y_n$, and $p > 1$

$$(\sum_{j=1}^n |x_j + y_j|^p)^{\frac{1}{p}} \leq (\sum_{j=1}^n |x_j|^p)^{\frac{1}{p}} + (\sum_{j=1}^n |y_j|^p)^{\frac{1}{p}}$$

Definition A subset $U \subset X$ of a metric space (X, d) is called open if, for every $x \in U$, there exists a constant $\epsilon > 0$ such that the open ball

$$B(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\}$$

(centered at x with radius ϵ) is contained in U . ^{就是在开球中的每个点都有以该点为圆心的开球}

我们也称此开球称为圆心 x 附近的临域.



Separable 可分的: A metric space is called separable if it contains a countable dense subset. ($\mathbb{R}^k$ is separable)

Definition A subset $E \subset X$ of a metric space (X, d) is open if $E^c = X \setminus E$ is closed.

稠密集: 所有稠密子集的并集在全集中稠密.

Facts: ① Every ball $B(x, r)$ is open, for if $y \in B(x, r)$ then $B(y, r-s) \subset B(x, r)$

$\forall x \in X, \varepsilon > 0, B(x, \varepsilon) \cap A \neq \emptyset$. A is a dense subset of X , i.e. A is dense in X .

② X and $\emptyset$ are both open.

③ 开集的并集仍开

有限

④ The intersection of any finite family of open sets is open.

开的交还是开, 开也是

↑ The union of any finite family of closed is closed.

Proof: $U_1, \dots, U_n$ are open, $x \in \bigcap_{i=1}^n U_i$. 闭的并还是闭

For each j there exists $r_j > 0$ such that $B(x, r_j) \subset U_j$

And then $B(x, r) \subset \bigcap_{i=1}^n U_i$ where $r = \min(r_1, \dots, r_n)$.

Definition A point $x \in E$ is said to be an interior point of E if

$\exists r > 0$, s.t. $B(x, r) \subset E$.

① ② $E = E^\circ$

The interior of E is the set of interior points denoted by E° .

A point x (not in E) is an exterior point of E when

$\exists r > 0$, s.t. $B(x, r) \subset X \setminus E$

$x \in \partial E \Leftrightarrow \forall B(x, r) \cap E \neq \emptyset \&$

The set of exterior points denoted by $(E)^\circ$.

$\forall B(x, r) \cap E^c \neq \emptyset$

All other points are called boundary points ∂E .

The set of interior and boundary points of E is called the closure of E

and denoted by $\bar{E} = E^\circ \cup \partial E$. $\bar{E}$ is also the intersection of all closed sets containing E .

Dense 稠密, nowhere dense 无处稠密的: E is dense in X if $\bar{E} = X$, and

nowhere dense if E has empty interior. eg. $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is n.d.

Definition Let X be a metric space. Let $E \subset X, x_0 \in X$.

i) x_0 is an adherent point of E if for every $r > 0$, $B(x_0, r) \cap E \neq \emptyset$;

触点

The set of all adherent points of E is just the closure of E .

ii) x_0 is a limit point of E if for every $r > 0$, $B(x_0, r) \cap (E \setminus \{x_0\}) \neq \emptyset$.

限制 $\rightarrow$ 在 E 中: closed $E' \subseteq E$

iii) E is closed if every limit point of E is a point of E .

iv) E is perfect if E is closed and if every point of E is a limit point of E .
 完全集 | 完备集
 在 E 中 $\rightarrow$ 限制点, perfect

Theorem If p is a limit point of a set E , then for any $r > 0$, $B(p, r)$ contains infinitely many points of E . (若只包含有限的点, 令 $q_1, \dots, q_n$ 为 $B(p, r) \cap E$ 除 p 外的点, 令 $s = \min_{1 \leq i \leq n} d(p, q_i)$, $B(p, s) \cap (E \setminus \{p\}) = \emptyset$, 那么 p 不是 E 的极限点, 矛盾了)
 极限点的开球包含无限 E 中的点

Corollary 推论: A finite point set has no limit points

Theorem A set E is open if and only if its complement is closed

($\Leftarrow$ E^c closed. Choose $x \in E$, 那么 $x \notin E^c$ 并且 x 不是 E^c 的极限点, 于是存在 $B(x, r) \cap E^c = \emptyset$, 即 $B(x, r) \subseteq E$, 那么 E is open.

$\Rightarrow E$ open, Let x 是 E^c 的极限点, 那么 $B(x, r)$ 一定包含 E^c 的点, 故 x 不是 E 的内点 $\xrightarrow{E \text{ is open}} x \in E^c \xrightarrow{E^c \text{ 的 l.p. 都在 } E^c \text{ 中}} E^c \text{ closed.}$)

Definition If X is a metric space, $E \subseteq X$, E' denotes the set of all limit points of E in X , then the closure of E is the set $\bar{E} = E \cup E'$

Theorem If X is a metric space and $E \subseteq X$, then

(a) $\bar{E}$ is closed

$$B(p, r) \subseteq (E^c)^c$$

(If $p \in X$ and $p \notin \bar{E} = E \cup E'$, hence $B(p, r) \cap \bar{E} = \emptyset$. The complement of $\bar{E}$ is therefore open. Hence $\bar{E}$ is closed.

(b) $\bar{E} = E$ if and only if E is closed.

(Because $\bar{E}$ is closed, $\bar{E} = \bar{\bar{E}} \Rightarrow E$ is closed; If E is closed, $E' \subseteq E$, hence $\bar{E} = E$)

(c) $\bar{E} \subseteq F$ for every closed set $F \subseteq X$ such that $E \subseteq F$

证: $\bar{E} = E \cup E' = E^\circ \cup \partial E =$ the set of all adherent points

$$\Rightarrow E \cup E' \subseteq E \cup \partial E \quad \text{也可写成 } E \cup E' = E \cup \partial E$$

$$\forall x \in E', \text{ 有 } x \notin E, \text{ aim: } x \in \partial E$$

$$x \in B(x, r), x \notin E \rightarrow B(x, r) \cap E^c \neq \emptyset$$

Hence $x \in \partial E$

$$\hookrightarrow \bar{E} \cup \partial E \subset E \cup E'$$

$$\forall x \in \partial E, \exists x \notin E \text{ (aim: } x \in E')$$
$$\therefore B(x, r) \cap E \neq \emptyset, x \notin E$$

$$\therefore B(x, r) \cap (E \setminus \{x\}) = B(x, r) \cap E \neq \emptyset$$

Hence $x \in \bar{E}'$.

Prop. $\textcircled{1}$ $U \subset X$ is open $\Leftrightarrow U$ is the union of a family of open balls.

② $Y \subset X$, $V \subset Y$ is open in $Y \Leftrightarrow \exists$ open set $U \subset X$ such that $V = U \cap Y$

Definition Let E be a subset of $\mathbb{R}$, and M be a real number. We say that M is a **least upper bound** for E iff (a) M is an upper bound for E , and also (b) any other upper bound M' for E must be larger than or equal to M .

The least upper bound of a set E is denoted by $\sup E$.

Theorem. Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in E$. Hence $y \in E$ if E is closed.

Definition Suppose $E \subset Y \subset X$, where X is a metric space. We say that E is open relative to Y if to each $p \in E$ there is an $r > 0$ such that

$$B(p, r) \cap Y \subset E$$

Theorem. Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X .

Definition By an open cover of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

Definition A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

↓ $K \subset G_1, U \dots UG_n$, 先属于可不行, 然后从 1 到 n 列出来

Only finite sets are compact.

Theorem (1) Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y .

($\Leftarrow$ Suppose K is compact relative to X , and let $\{V_\alpha\}$ be open sets of Y , s.t. $K \subset \bigcup_\alpha V_\alpha$. There open sets U_α of X , such that $V_\alpha = Y \cap U_\alpha$, for $\forall \alpha$; and since K is a compact set of X , $K \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$. combining with $K \subset Y$, implies that $K \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$. Thus K is compact relative to Y .

Suppose K is compact relative to Y , and let $\{U_\alpha\}$ be open sets of X s.t. $K \subset \bigcup_\alpha U_\alpha$, and let $V_\alpha = Y \cap U_\alpha$. Because $K \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$, $K \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$. Thus K is compact relative to X .)

Theorem (2) Compact subsets of metric spaces are closed.

(即证 Y^c open. First fix $x \in Y^c$, $\therefore \forall y \in Y, r_y = d(x, y) > 0$

$B(x, \frac{r_y}{2}) \cap B(y, \frac{r_y}{2}) = \emptyset$, 且 $\{B(y, \frac{r_y}{2})\}_{y \in Y}$ 是 Y 的一个 open cover. 开覆盖

$\therefore Y$ 紧, $\therefore \exists y_1, \dots, y_n \in Y$ s.t. $\bigcup_{i=1}^n B(y_i, \frac{r_{y_i}}{2}) \supset Y$

Let $r = \min \frac{r_{y_i}}{2}$, then $B(x, r) \cap B(y_i, \frac{r_{y_i}}{2}) = \emptyset$ ($i=1, \dots, n$)

Thus $B(x, r) \cap Y = \emptyset \rightarrow B(x, r) \subset Y^c \rightarrow Y^c$ is open.)

Theorem (3) Closed subsets of compact sets are compact.

(提示: If F is closed and K is compact, then $F \cap K$ is compact.

Let (X, d) be compact, $Y \subset X$ is closed $\Rightarrow Y$ is a compact subset of X .

pf: Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of Y (U_α is open in X , $\bigcup_{\alpha \in A} U_\alpha \supset Y$)

$\rightarrow Y^c \cup (\bigcup_{\alpha \in A} U_\alpha) \supset Y^c \cup Y = X$, Y^c is open in X

Because X is compact, $\exists Y^c, U_{\alpha_1}, \dots, U_{\alpha_n}$ such that $Y^c \cup (\bigcup_{i=1}^n U_{\alpha_i}) = X$

$\rightarrow (Y^c \cup (\bigcup_{i=1}^n U_{\alpha_i})) \cap Y = Y \rightarrow (\bigcup_{i=1}^n U_{\alpha_i}) \cap Y = Y \rightarrow \bigcup_{i=1}^n U_{\alpha_i} \supset Y$

Hence, Y is compact.

Theorem (4) If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty,

then $\bigcap K_n \neq \emptyset$.

(推论: If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$, $n=1, 2, \dots$, then $\bigcap K_n \neq \emptyset$.)

pf: If $\bigcap K_n = \emptyset$, let $K_{a_1}, K_{a_2}, \dots, K_{a_n}, \dots$ such that $K_{a_n} \cap (\bigcap_{a \neq a_n} K_a) = \emptyset \Rightarrow K_{a_n} \subset (\bigcap_{a \neq a_n} K_a)^c$
 $\Rightarrow K_{a_n} \subset \bigcup_{a \neq a_n} K_a^c$

$\{K_a\}$ are compact subsets in $(X, d) \Rightarrow K_a$ is closed $\Rightarrow K_a^c$ is open

$\because K_{a_1}$ is compact, $\exists a_1, \dots, a_n$ s.t. $K_{a_1} \subset K_{a_1}^c \cup \dots \cup K_{a_n}^c$

$\Rightarrow K_{a_1} \cap (K_{a_1}^c \cup \dots \cup K_{a_n}^c) = \emptyset \Rightarrow K_{a_1} \cap (\bigcap_{a \neq a_1} K_a) = \emptyset \Rightarrow K_{a_1} \cap \dots \cap K_{a_n} = \emptyset$

Which contradicts to "every finite subcollection's intersection is nonempty"

Theorem (5) If $\{I_n\}$ is a sequence of finite closed intervals in $\mathbb{R}$ such that $I_n \supset I_{n+1}$ ($n=1, 2, 3, \dots$), then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

(Let $I_n = [a_n, b_n]$, $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$)

$\because \{a_n\}$ 单增且 $a_n \leq b_1$, $\therefore$ 极限存在, $\lim_{n \rightarrow \infty} a_n = x$

Since $a_m \leq x \leq b_m$, $x \in I_m$ for $m=1, 2, 3, \dots$

Thus $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

Theorem (6) Let k be a positive integer. If $\{I_n\}$ is a sequence of k -cells such that $I_n \supset I_{n+1}$ ($n=1, 2, 3, \dots$), then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

(k -cells 可以看作 k 个正方形, 由大到小圈起来)

Theorem (7) Every k -cell is compact. (假设了不能有限集完全 cover)

(Suppose $\exists$ an open cover $\{G_\alpha\}$ of I which contains no finite subcover of I . Put $C_j = (a_j + b_j)/2$, the intervals $[a_j, C_j]$ and $[C_j, b_j]$ determine 2^k k -cells Q_i whose union is I . At least one of these sets Q_i , call it I_1 , cannot be covered by any finite subcollection of $\{G_\alpha\}$.

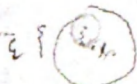
We next subdivide I_1 and continue the process. The sequence $\{I_n\}$ satisfies

(a) $I_1 \supset I_2 \supset \dots$,

(b) I_n is not covered by any finite subcollection of $\{G_\alpha\}$;

No.

Date

 $k \geq 1$ $k=1$ 

在度量空间中, X compact, 任何序列
都有收敛子列, 且收敛到 X 内.

X is a metric space in which every
infinite subset has a limit point ^{bounded}
不用在 $\mathbb{R}^k$ 上 $\downarrow \rightarrow \textcircled{X}$ is totally

$\cup X$ is separable. $\textcircled{X}$ is compact.

compact $\rightarrow$ countable base $\Leftrightarrow$ separable

(c) if $x, y \in I_n$, then $|x - y| \leq 2^{-n} \delta$.

totally bounded $\rightarrow$ separable

$\exists x^* \in I_n$. For some b , $x^* \in C_b$.

$\wedge X$ can be covered by finitely many
neighborhoods of radius δ . 有限覆盖

Because C_b is open, $\exists r > 0$ s.t. $|y - x^*| < r \rightarrow y \in C_b$. If $2^{-n} \delta < r$. (c) $\rightarrow$
 $I_n \subset C_b$, contradicts (b).

Theorem (8) Let E be a subset of $(\mathbb{R}^k)$. The following conditions are equivalent.

修改的: (a) E is closed and bounded. (d) 是 compact set 的闭子集
等价条件 (b) E is compact. (e) E is

(c) Every infinite subset of E has a limit point in E .

(proof: $\because E$ is bounded $\therefore E \subset I$ for some k -cell I .

$\therefore k$ -cell is compact & closed subset of compact set is compact.

$\therefore E$ is compact

If $\exists$ an infinite subset Y of E s.t. any point of E is not the
limit point of Y .

$\forall x \in E, \exists r_x > 0, B(x, r_x) \cap (Y - \{x\}) = \emptyset$

$\rightarrow B(x, r_x)$ 最多只包含一个 Y 中的点

$\therefore Y$ 是一个无限集

$Y \subset E \rightarrow$ 不能用有限个 open set $B(x, r_x)_{x \in E}$ 包含住 Y
与 E compact 冲突

故 Y has a limit point in X .

如果 E 不有界. Let E contains infinitely many points x_k with
 $|x_k| > n_k$ ($k=1, 2, 3, \dots$). where $n_1 < n_2 < \dots \rightarrow \infty$ 发散到无穷

故 E 无极限点 (每二者之间都可以找到半径比距离小的圆, 不

Contradicts (c) 像有极限的数列, 在极限处取开圆, 一定包含其它点)

如果 E 不闭, 则存在 $x_0 \in \mathbb{R}^k$ s.t. $x_0 \in E'$ 但 $x_0 \notin E$

For $n=1, 2, 3, \dots, \exists x_n \in E$ s.t. $|x_n - x_0| < \frac{1}{n}$ 可以任意小, 满足极限的定义

$S = \{x_n\}$, S is an infinite set. 否则 $\exists r$ s.t. $B(x_0, r) \cap E = \emptyset$

Hence, x_0 is also infinite subset S 's limit point.

Then, we want to show S has no other limit point in $\mathbb{R}^k$.

For $y \in \mathbb{R}^k$, $y \neq x_0$, $|x_n - y| \geq |x_0 - y| - |x_n - x_0| = |x_0 - y| - \frac{1}{n} \geq \frac{1}{2}|x_0 - y|$

where $n \rightarrow \infty$, $B(y, \frac{1}{2}) \cap \{x_1, \dots, x_n\}$ 是有限集

故 y 不是 S 的极限点

$\therefore x_0 \notin E$

$\therefore$ contradicts $\rightarrow E$ 是闭集

相当于分别以 x_0, y 为圆心画两个球，因为已知 x_0 为一个极限点，所以它

Theorem (Weierstrass) Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Definition Two subsets A and B of a metric space X are said to be **separated** if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty. A set $E \subset X$ is **connected** if E is not a union of two nonempty separated sets.

注: Separated sets are disjoint, but disjoint sets needn't be separated.

$[0, 1]$ & $(1, 2)$ Not separated, $(0, 1)$ & $(1, 2)$ separated

Theorem A subset E of $\mathbb{R}$ is **connected** if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

Theorem (9) A nonempty perfect set P in $\mathbb{R}^k$ is uncountable. $\mathbb{R}^k$ 中的完美集

(P has limit points $\rightarrow P$ is infinite. 不可数)

Assume P is countable, $P = \{x_1, x_2, \dots\}$

Construct a sequence $\{V_n\}$:

V_1 是 x_1 的邻域: $\therefore x_1$ is a limit point $\rightarrow V_1 \cap \{x_1\} \cap P \neq \emptyset$ 至少还存在一点.

选取 $V_2 \subset V_1$, $x_1 \notin V_2$, $V_2 \cap P \neq \emptyset$

↓ 递推, we can find neighborhoods V_{n+1} s.t.

i) $V_{n+1} \subset V_n$, ii) $x_n \notin V_{n+1}$, iii) $V_{n+1} \cap P \neq \emptyset$.

$\therefore V_n$ is bounded and closed $\rightarrow V_n$ is compact.

$\therefore$ Set $K_n = \bar{V}_n \cap P$ $\xrightarrow{V_n \text{ compact, } P \text{ closed}}$ K_n compact and nonempty.

$K_1 \supset K_2 \supset \dots \Rightarrow \bigcap_{n=1}^{\infty} K_n \neq \emptyset$

$x_{n+1} \notin K_n \rightarrow \forall n, x_n \notin K \rightarrow P \cap K = \emptyset$. But $K \subset P$

Contradicts. 关键在这, 如果可列那么完全可以列举完

定义

Definition A point x in a metric space X is said to be a limit of a sequence of points $(x_n) \subset X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

If a sequence has a Δ limit, it is called convergent.

bounded (有界的): A subset Y of a metric space (X, d) is bounded if there exists $x \in X$ and $r > 0$, s.t. $Y \subset B(x, r)$. Otherwise, Y is unbounded. (被包含?)

柯西
Cauchy 准则: (x_n) converges $\Leftrightarrow (x_n)$ is Cauchy i.e. $\forall \varepsilon > 0, \exists N$ s.t. $|x_n - x_m| < \varepsilon, \forall m, n \geq N$ 柯西序列的定义

Lemma 1. Let X be a metric space.

i) If $p \in X, p' \in X$, and if $\{p_n\}$ converges to p and to p' , then $p = p'$.

ii) If $E \subset X$ and if p is a limit point of E , then there is a sequence $\{p_n\}$ in E s.t. $p_n \rightarrow p$.

(pf: " $|d(p, p')| \leq d(p_n, p) + d(p_n, p')$ 极限点与数列极限的关系 利用极限的定义)

As $n \rightarrow \infty, d(p, p') \leq 0 + 0 = 0 \rightarrow d(p, p') = 0$. Thus $p = p'$ 把两球换成距离 绝对值

(ii) For each positive integer n , there $\exists$ a point $p_n \in E$ s.t. $d(p_n, p) < \frac{1}{n}$.

Hence $p_n \rightarrow p$ 这样就变成了

Theorem Suppose $x_n \in \mathbb{R}^k (n = 1, 2, 3, \dots)$ and $x_n = (a_{1,n}, \dots, a_{k,n})$. Then $\{x_n\}$ converges to $x = (a_1, \dots, a_k)$ if and only if 分量收敛

$$\lim_{n \rightarrow \infty} a_{j,n} = a_j \quad (1 \leq j \leq k) \quad \text{分量收敛, 点收敛}$$

($\Rightarrow |a_{j,n} - a_j| \leq |x_n - x|$ 分量差 < 总差)

$\Leftarrow$ 构造法. For $\forall \varepsilon > 0, \exists N$, s.t. $|a_{j,n} - a_j| < \frac{\varepsilon}{\sqrt{k}}$ for $n \geq N$

$$\rightarrow |x_n - x| = \left(\sum_{j=1}^k |a_{j,n} - a_j|^2 \right)^{\frac{1}{2}} < \varepsilon$$

$$\rightarrow x_n \rightarrow x \quad \text{极限点与开集}$$

Lemma 2. $x_n \rightarrow x \Leftrightarrow \forall$ open set U that contains x there $\exists$ an N s.t. $x_n \in U$ for $n \geq N$.

($\Rightarrow$ Given an open set U contains $x \rightarrow \exists B(x, r) \subset U$ for $r > 0$.)

Hence, there $\exists N > 0$, s.t. $x_n \in B(x, r) \subset U$ for $n \geq N$.

$\Leftarrow \forall \varepsilon > 0, \exists N > 0$, s.t. $x_n \in B(x, \varepsilon)$ for $n \geq N \rightarrow d(x, x_n) < \varepsilon \rightarrow x_n \rightarrow x$)

X is compact $\Leftrightarrow X$ is complete and totally bounded.

紧性仅取决于拓扑, 有界性则取决于度量.

怎么证明

Lemma 3. A subset E of (X, d) is closed $\Leftrightarrow$ whenever $(x_n) \subset E$ with $x_n \rightarrow x$ it follows that $x \in E$. 极限点与闭集

($\Rightarrow$) Let $(x_n) \subset E$ with $x_n \rightarrow x$. If $x \notin E \rightarrow x \in E^c$.

$\therefore E^c$ is open $\therefore \exists B(x, r) \subset E^c$ for $r > 0 \rightarrow B(x, r) \cap E = \emptyset$

Contradicts to $x_n \rightarrow x$

$\Leftarrow$ Aim: $E' \subset E$

因为在那个特殊的开球内

$\forall x \in E'$ (根据 limit point 定义) $\rightarrow \exists (x_n) \subset E$, with $x_n \rightarrow x \xrightarrow{E \subset E} x \in E$

Hence $E' \subset E$, E is closed.

Definition Subsequence (子序列), 指从最初序列通过除去某些元素但不破坏余下元素的相对位置形成的新序列. $\{p_{n_k}\}$. If $\{p_{n_k}\}$ converges, its limit is

called a subsequential limit of $\{p_n\}$. 若完备空间中序列收敛至 x , $x \in X$

注: $\{p_n\} \rightarrow p$ iff every subsequence of $\{p_n\} \rightarrow p$. 子序列收敛

Complete 完备的: A metric space (X, d) is called complete if every Cauchy sequence in X converges. (For any $n \in \mathbb{N}$, $\mathbb{R}^n$ is complete; all compact metric spaces are complete)

Theorem (a) In any metric space X , every convergent sequence is a Cauchy sequence. 可以由收敛推 Cauchy, 不管空间是否完备

(b) Let (x_n) be a Cauchy sequence in (X, d) . If $\exists (x_{n_k}) \subset (x_n)$ and $x \in X$ with $x_{n_k} \xrightarrow{E \subset E} x$, then $x_n \xrightarrow{E \subset E} x$. 性质良好: 可从单个直接推广到所有

((a) Let $x_n \in X$, $\forall \varepsilon > 0$, $\exists N$ s.t. $d(x_n, x) < \frac{\varepsilon}{2}$ for $n \geq N$.

If $n, m \geq N$, $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \varepsilon$

Hence, (x_n) is Cauchy

(b) $\forall \varepsilon > 0$

$\therefore (x_n)$ is Cauchy

$\therefore \exists N_1$ s.t. $d(x_n, x_m) < \frac{\varepsilon}{2}$, for $m, n \geq N_1$

$\therefore x_{n_k} \rightarrow x$, $\exists K$ s.t. $d(x_{n_k}, x) < \frac{\varepsilon}{2}$, for $k \geq K$

Let $N = \max(N_1, K)$ $N \geq N_1$ and $N \geq K$ N 和 N_1 ?

If $n \geq N$, $d(x_n, x) \leq d(x_n, x_{n_N}) + d(x_{n_N}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Hence, $x_n \rightarrow x$ $n_N \geq N$ 成立, 不需要证明.

Topological bases 拓扑基. A collection $\{V_\alpha\}$ of open subsets of X is said to be a base for X if, for every $x \in X$ and every open set $G \subset X$ s.t. $x \in G$, we have $x \in V_\alpha \subset G$ for some α . In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Theorem

Let (X, d) be a metric space. The following are equivalent.

- i) X is compact.
- ii) Any sequence in X has a convergent subsequence.
- iii) X is complete and totally bounded.

pf: (i) $\rightarrow$ (ii) We can find a infinite subsequence C this sequence.

Any infinite subsequence has a limit point $\rightarrow$ it is convergent.



$$B(x_{n_k}, \frac{1}{n_k}) \subset B(x_0, \frac{1}{n}) \subset U_\alpha$$

练习

Numerical Sequences and Series 数列和级数

Definition The sequence $\{a_n\}$ converges to the number L if for every positive number ε there corresponds an integer N such that

$$|a_n - L| < \varepsilon \text{ whenever } n > N$$

Definition The sequence $\{a_n\}$ diverges to infinity if for every number M there is an integer N such that for all n larger than N , $a_n > M$. If this condition holds we write

$$\lim_{n \rightarrow \infty} a_n = \infty \text{ or } a_n \rightarrow \infty$$

If for every number m there is an integer N such that for all $n > N$ we have $a_n \leq m$, then we say $\{a_n\}$ diverges to negative infinity and write

$$\lim_{n \rightarrow \infty} a_n = -\infty \text{ or } a_n \rightarrow -\infty$$

Definition A sequence $\{S_n\}$ of real numbers is said to be

(a) monotonically increasing if $S_n \leq S_{n+1}$ ($n=1, 2, 3, \dots$);

(b) monotonically decreasing if $S_n \geq S_{n+1}$ ($n=1, 2, 3, \dots$).

Definition

(a) If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the least upper bound for $\{a_n\}$.

(b) If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the greatest lower bound for $\{a_n\}$.

Theorem Suppose $\{S_n\}$ is monotonic. Then $\{S_n\}$ converges if and only if it is bounded.

proof: Suppose $S_n \leq S_{n+1}$ (单增的证明类似)

$\Rightarrow$ Suppose $\lim_{n \rightarrow \infty} S_n = L$. because $\{S_n\}$ is monotonically increasing $\rightarrow S_n < L+1$

$\because S_n \geq S_1 \rightarrow S_n \in [S_1, L+1]$ for $n \in \mathbb{N}^* \rightarrow \{S_n\}$ is bounded.

$\Leftarrow$ If $\{S_n\}$ is bounded, let S be the least upper bound of $\{S_n\}$.

$\rightarrow S_n \leq S$ for $n \in \mathbb{N}^*$

No.

Date

For every $\varepsilon > 0$, there is an integer N such that $S - \varepsilon < S_n \leq S$, for otherwise $S - \varepsilon$ would be an upper bound of E .

$\{S_n\}$ increases, if $n > N \rightarrow S - \varepsilon < S_n \leq S$

Hence, $\{S_n\}$ converges (to S).

Definition (Limit superior 上极限 and limit inferior 下极限) Suppose that $\{a_n\}_{n=m}^{\infty}$ is a sequence. We define a new sequence $(a_n^+)_{n=m}^{\infty}$ by the formula

$$a_n^+ := \sup(a_n)_{n=n}^{\infty} = \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}$$

i.e. a_n^+ is the supremum of all the elements in the sequence from a_n onwards.

We then define the limit superior of the sequence $\{a_n\}_{n=m}^{\infty}$, denoted $\limsup_{n \rightarrow \infty} a_n$, by the formula

$$\limsup_{n \rightarrow \infty} a_n := \inf(a_n^+)_{n=m}^{\infty}$$

Similarly, we can define

$$a_n^- := \inf(a_n)_{n=n}^{\infty}$$

and define the limit inferior of the sequence $\{a_n\}_{n=m}^{\infty}$, denoted $\liminf_{n \rightarrow \infty} a_n$, by the formula

$$\liminf_{n \rightarrow \infty} a_n := \sup(a_n^-)_{n=m}^{\infty}$$

eg1. Let $a_1, a_2, a_3, \dots$ denote the sequence $1, -1, 1, -1, 1, -1, 1, -1, \dots$

Then $a_1^+, a_2^+, a_3^+, \dots$ is the sequence $1, 1, 1, 1, 1, 1, 1, 1, \dots$

which infimum is 1. Hence, the limit superior is 1.

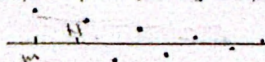
Then $a_1^-, a_2^-, a_3^-, \dots$ is the sequence $-1, -1, -1, -1, -1, -1, -1, -1, \dots$

which supremum is -1 . Hence, the limit inferior is -1 .

注: 上极限不一定等于最大值 (supremum), 下极限不一定等于最小值 (infimum).

eg2. Let $a_1, a_2, a_3, \dots$ denote the sequence $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \dots$

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目前想的是 $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = L$, 则 $\lim_{n \rightarrow \infty} a_n = L$ (极限存在且三者相等, 充要条件: 单增或有界且有下确界, 所以它有极限).

$$a_1^+, a_2^+, \dots \rightarrow 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \rightarrow \limsup_{n \rightarrow \infty} a_n = 0$$

$$a_1^-, a_2^-, \dots \rightarrow -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, -\frac{1}{6}, \dots \rightarrow \liminf_{n \rightarrow \infty} a_n = 0$$

eg 3. Let $a_1, a_2, a_3, \dots$ denote the sequence 1, 2, 3, 4, 5, 6, ... 所以它可以看成 $\{a_n\}$ 和其它 $\{a_n\}$ 组成的, 对于 $\{a_n\}$ 而言, 因为它收敛到 $\limsup_{n \rightarrow \infty} a_n$, 故存在 $N \geq m$, 使得 $a_n^+ < \limsup_{n \rightarrow \infty} a_n + \varepsilon$ for $n \geq N$. 对于非 $\{a_n^+\}$ 中的元素而言 $a_n < a_n^+ < \limsup_{n \rightarrow \infty} a_n + \varepsilon$

$$a_1^+, a_2^+, \dots \rightarrow +\infty, +\infty, +\infty, \dots \rightarrow \limsup_{n \rightarrow \infty} a_n = +\infty$$

$$a_1^-, a_2^-, \dots \rightarrow 1, 2, 3, 4, \dots \rightarrow \liminf_{n \rightarrow \infty} a_n = +\infty$$

证: $\inf_{n \geq m} (a_n)_{n \geq m} \leq L^- \leq L^+ \leq \sup_{n \geq m} (a_n)_{n \geq m}$.
最小值 下确界 上确界 最大值

Theorem (a) If $p > 0$, then $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$

(b) If $p > 0$, then $\lim_{n \rightarrow \infty} \sqrt[p]{p} = 1$

(c) $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$

(d) If $p > 0$ and α is real, then $\lim_{n \rightarrow \infty} \frac{n^\alpha}{(1+p)^n} = 0$

(e) If $|x| < 1$, then $\lim_{n \rightarrow \infty} x^n = 0$

proof of (d): 二项式定理 $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$

Let k be an integer s.t. $k > \alpha$, $k > 0$. For $n > 2k$

$$(1+p)^n = \sum_{i=0}^n \binom{n}{i} p^i > \binom{n}{k} p^k = \frac{n(n-1)\dots(n-k+1)}{k!} p^k > \frac{n^k}{2^k k!} p^k$$

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k}$$

Because $\alpha - k < 0$, $n^{\alpha-k} \rightarrow 0$ by (a).

夹逼: $\frac{n^\alpha}{(1+p)^n} \rightarrow 0$

Theorem Let $L = \limsup_{n \rightarrow \infty} a_n$.

i) $\exists (a_{n_k}) \subset (a_n)$ with $\lim_{k \rightarrow \infty} a_{n_k} = L$ 上极限

ii) If $(a_{n_k}) \subset (a_n)$ with $\lim_{k \rightarrow \infty} a_{n_k} = X$, then $X \leq L$. 子序列问题

(pf: i) For $L+1$, $\exists N_1$ s.t. $a_n < L+1$, $\forall n \geq N_1$.

For $L-1$, for this N_1 , $\exists n_1 \geq N_1$ s.t. $a_{n_1} > L-1$.

$$\rightarrow L-1 < a_{n_1} < L+1 \rightarrow L-\frac{1}{2} < a_{n_1} < L+\frac{1}{2}$$

We can find a subsequence (a_{n_k}) of (a_n) s.t. $L-\frac{1}{k} < a_{n_k} < L+\frac{1}{k}$

Hence, $\lim_{k \rightarrow \infty} a_{n_k} = L$.

或. exist 反例

If $L = +\infty$, $\{a_n^*\}$ doesn't bounded above $\rightarrow \{a_n\}$ doesn't bounded

$\therefore \exists (a_{n_k}) \subset (a_n)$ with $a_{n_k} \rightarrow +\infty$

ii) Suppose $\exists (a_{n_k}) \subset (a_n)$ with $\lim_{k \rightarrow \infty} a_{n_k} = X > L$

$\forall n \in \mathbb{N}^*$, $\exists N_k \geq n$ s.t. $a_{N_k} > L + \frac{\epsilon}{2}$ let $X = L + \epsilon$

$$a_n^* = \sup \{a_n, a_{n+1}, \dots, a_{N_k}, a_{N_k+1}, \dots\} \geq a_{N_k} > L + \frac{\epsilon}{2}$$

$\therefore \lim_{n \rightarrow \infty} a_n^* \geq L + \frac{\epsilon}{2}$ 因为左也是指下界, 而右边数列又是单减的

$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n^* \geq L + \frac{\epsilon}{2}$. Contradiction

Proposition Let $L = \limsup_{n \rightarrow \infty} a_n$, then

i) If $x > L$, then $\exists N$ s.t. $a_n < x$, $\forall n \geq N$ 上极限

ii) If $x < L$, $\forall N \in \{1, 2, \dots\}$, $\exists n \geq N$ s.t. $a_n > x$

(pf: i) Suppose there $\exists a_{n_k} > x$, ($n_k > n \geq N$)

$$\forall n, a_n^* = \sup \{a_n, a_{n+1}, \dots, a_{n_k}, \dots\} \geq x$$

$$\rightarrow \lim_{n \rightarrow \infty} a_n^* \geq x > L. \text{ Contradiction}$$

ii) Otherwise, $\exists N$ s.t. if $n \geq N$, then $a_n \leq x$

$$a_n^* = \sup \{a_n, a_{n+1}, \dots\} \leq x \text{ for } n \geq N.$$

$$\lim_{n \rightarrow \infty} a_n^* \leq x < L. \text{ Contradiction.}$$

Theorem $\sum a_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that $|\sum_{k=n}^m a_k| \leq \varepsilon$ if $m \geq n \geq N$. 柯西序列的应用

Theorem (比较检验 - 直接比较)

(a) If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.

(b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

Theorem (积分检验)

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x) dx$ both converge or both diverge.

Theorem (p -级数)

$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \dots + \frac{1}{n^p} + \dots$ converges if $p > 1$, diverges if $p \leq 1$.

推广: $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ or $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if $p > 1$, diverges if $p \leq 1$

$$\text{pf: } \int_2^{\infty} \frac{1}{x(\log x)^p} dx = \int_2^{\infty} \frac{(1/n)^p}{(\ln x)^p} d(\ln x), \quad \int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \int_2^{\infty} \frac{1}{(\ln x)^p} d(\ln x)$$

Theorem (几何级数)

If $|r| < 1$, the geometric series $a + ar + ar^2 + \dots + ar^{n-1} + \dots$ converges to $a/(1-r)$: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, $|r| < 1$. If $|r| \geq 1$, the series diverges.

Definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. Here $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$ if $n \geq 1$, and $0! = 1$.

$$S_n = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots + \frac{1}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} < 3$$

Theorem e is irrational.

极限是3, 故不取等

(pf: Assume $\exists p, q \in \mathbb{N}^*$ s.t. $e = 1 + \frac{1}{2} + \frac{1}{3} + \dots = \frac{q}{p}$ ($p \neq 1$))

$$p! + \frac{p!}{2!} + \dots + \frac{p!}{p!} + \frac{p!}{(p+1)!} + \dots = q(p-1)!$$

$$\text{Because } (p! + \frac{p!}{2!} + \dots + \frac{p!}{p!}), q(p-1)! \in \mathbb{Z}^* \rightarrow \frac{p!}{(p+1)!} + \frac{p!}{(p+2)!} + \dots \in \mathbb{Z}^*$$

$$\text{But } \frac{p!}{(p+1)!} + \frac{p!}{(p+2)!} + \dots = \frac{1}{p+1} + \frac{1}{(p+1)(p+2)} + \dots < \frac{1}{p+1} + \frac{1}{(p+1)^2} + \dots = \frac{\frac{1}{p+1}}{1 - \frac{1}{p+1}} = \frac{1}{p} < 1$$

Theorem (根值法)

Given $\sum a_n$, put $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$. Then

(a) if $\alpha < 1$, $\sum a_n$ converges;

(b) if $\alpha > 1$, $\sum a_n$ diverges;

(c) if $\alpha = 1$, the test gives no information.

(pf: (a) If $\alpha < 1$, choose β s.t. $\alpha < \beta < 1$. Because $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$,

$\exists N$ s.t. $\sqrt[n]{|a_n|} < \beta$ for $n \geq N$.

For $n \geq N$, $|a_n| < \beta^n$. $\sum \beta^n$ converges $\rightarrow \sum a_n$ converges

(b) If $\alpha > 1$, $\exists (a_{n_k}) \subset (a_n)$ with $\sqrt[n_k]{|a_{n_k}|} \rightarrow \alpha$

Hence $|a_n| > 1$ for infinitely many values of n , $a_n \rightarrow 0$ does not hold.

(c) Example: $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$

Theorem (极限比较法)

The series $\sum a_n$

(a) converges if $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$,

(b) diverges if $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.

Theorem For any sequence $\{C_n\}$ of positive numbers,

$$\liminf_{n \rightarrow \infty} \frac{C_{n+1}}{C_n} \leq \liminf_{n \rightarrow \infty} \sqrt[n]{C_n} \leq \limsup_{n \rightarrow \infty} \sqrt[n]{C_n} \leq \limsup_{n \rightarrow \infty} \frac{C_{n+1}}{C_n}$$

(proof of the second inequality: Put $\alpha = \limsup_{n \rightarrow \infty} \frac{C_{n+1}}{C_n}$

Assume $\alpha < \infty$ and take any $\beta > \alpha$.

There $\exists N \in \mathbb{N}^*$ s.t. $\frac{C_{n+1}}{C_n} \leq \beta$ for $n \geq N \rightarrow C_{n+k+1} \leq \beta C_{n+k}$ ($k=0,1,\dots,p-1$)

Multiplying these inequalities, we obtain $C_{n+p} \leq \beta^p C_n \rightarrow C_n \leq C_N \beta^{-N} \cdot \beta^n$ ($n \geq N$)

$\rightarrow \sqrt[n]{C_n} \leq \sqrt[n]{C_N \beta^{-N} \cdot \beta^n} \rightarrow \limsup_{n \rightarrow \infty} \sqrt[n]{C_n} \leq \beta$

Since this is true for every $\beta > \alpha$, we have $\limsup_{n \rightarrow \infty} \sqrt[n]{C_n} \leq \alpha$.

Definition $\sum_{n=0}^{\infty} C_n z^n = C_0 + C_1 z + C_2 z^2 + \dots + C_n z^n + \dots$ is called a **power series**. 幂级数

The numbers C_n are called the coefficients of the series.

Theorem Given the power series $\sum C_n z^n$, put $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|C_n|}$, $R = \frac{1}{\alpha}$.

$$\sqrt[n]{|C_n z^n|} = \sqrt[n]{|C_n|} |z| < 1$$

(If $\alpha=0$, $R=+\infty$; if $\alpha=+\infty$, $R=0$.) Then $\sum C_n z^n$ converges if $|z|<R$, and diverges if $|z|>R$.

$$\lim_{n \rightarrow \infty} \sqrt[n]{|C_n z^n|} = |z| \lim_{n \rightarrow \infty} \sqrt[n]{|C_n|} = |z| \alpha < 1 \rightarrow |z| < \frac{1}{\alpha} = R$$

Definition A series $\sum A_n$ converges absolutely if the corresponding series of absolute values, $\sum |A_n|$, converges.

Theorem (绝对收敛法)

If $\sum_{n=1}^{\infty} |A_n|$ converges, then $\sum_{n=1}^{\infty} A_n$ converges.

pf: $|\sum_{k=1}^m A_k| \leq \sum_{k=1}^m |A_k|$ + Cauchy criterion

Definition A series that is convergent but not absolutely convergent is called conditionally convergent.

$$\text{eg: } \sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \dots$$

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Theorem $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$

(pf: Let $s_n = \sum_{k=0}^n \frac{1}{k!}$, $t_n = (1 + \frac{1}{n})^n$)

$$t_n = 1 + \sum_{k=1}^n \binom{n}{k} \left(\frac{1}{n}\right)^k = 1 + 1 + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \dots$$

$$= 1 + 1 + \frac{1}{2!} (1 - \frac{1}{n}) + \frac{1}{3!} (1 - \frac{1}{n})(1 - \frac{2}{n}) + \dots + \frac{1}{n!} (1 - \frac{1}{n})(1 - \frac{2}{n}) \dots (1 - \frac{n-1}{n})$$

$$\text{Hence } t_n \leq s_n \rightarrow \limsup_{n \rightarrow \infty} t_n \leq e$$

$$\text{Next, if } n \geq m, t_n \geq 1 + 1 + \frac{1}{2!} (1 - \frac{1}{n}) + \dots + \frac{1}{m!} (1 - \frac{1}{n}) \dots (1 - \frac{m-1}{n})$$

$$\text{Let } n \rightarrow \infty, m \text{ fixed, } \liminf_{n \rightarrow \infty} t_n \geq 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{m!} \rightarrow s_m \leq \liminf_{n \rightarrow \infty} t_n$$

$$\text{Letting } m \rightarrow \infty, e \leq \liminf_{n \rightarrow \infty} t_n$$

$$\text{Sandwich theorem } \rightarrow \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$$

Theorem Given two sequences $\{a_n\}, \{b_n\}$, put $A_n = \sum_{k=0}^n a_k$ if $n \geq 0$; put $A_{-1} = 0$.

Then, if $0 \leq p \leq q$, we have $\sum_{n=p}^q a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$.

$$\text{pf: } \sum_{n=p}^q a_n b_n = \sum_{n=p}^q (A_n - A_{n-1}) b_n = \sum_{n=p}^q A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Theorem (Dirichlet 判别法) $\sum_{n=1}^{\infty} a_n b_n$ converges if $\{a_n\}$ is a sequence of partial sums and $\{b_n\}$ is a sequence of terms.

Suppose (a) the partial sums A_n of $\sum A_n$ form a bounded sequence; (b) $b_0 \geq b_1 \geq \dots$;

数列部分收敛判别

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(c) $\lim_{n \rightarrow \infty} b_n = 0$. Then $\sum a_n b_n$ converges.

pf: Choose M s.t. $|A_n| \leq M$ for all n . Given $\varepsilon > 0$, $\exists N$ s.t. $b_n \leq \frac{\varepsilon}{2M}$

$$\begin{aligned} \text{For } N \leq p < q, \quad \left| \sum_{n=p}^q a_n b_n \right| &= \left| \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right| \\ &\leq M \left| \sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p \right| \\ &= 2M b_p \leq 2M b_N \leq \varepsilon \end{aligned}$$

Convergence follows from the Cauchy criterion.

Theorem (交错级数法)

Suppose

(a) $|C_1| \geq |C_2| \geq |C_3| \geq \dots$; (b) $C_{2m-1} \geq 0, C_{2m} \leq 0 (m=1, 2, 3, \dots)$; (c) $\lim_{n \rightarrow \infty} C_n = 0$

Then $\sum C_n$ converges.

Definition Given $\sum a_n$ and $\sum b_n$, we put $C_n = \sum_{k=0}^n a_k b_{n-k} (n=0, 1, 2, \dots)$ and call $\sum C_n$ the product of the two given series. **数列乘法定义: 交叉相乘再相加**

复合函数连续

Homeomorphism 同胚: T is a homeomorphism iff T is bijective and T and T^{-1} are continuous.

Theorem T is continuous $\Leftrightarrow \forall$ open $G \subset Y, T^{-1}(G)$ is open in X .

Proof: $\Rightarrow$ Let $G \subset Y$ be open, $x_0 \in T^{-1}(G) \rightarrow T(x_0) \in G \rightarrow \exists B(Tx_0, \varepsilon) \subset G$
 Because T is continuous, $\forall \varepsilon > 0, \exists \delta > 0$, s.t. $d(x, x_0) < \delta \rightarrow d(Tx, Tx_0) < \varepsilon$
 $\rightarrow T(B(x_0, \delta)) \subset B(Tx_0, \varepsilon) \rightarrow B(x_0, \delta) \subset T^{-1}(B(Tx_0, \varepsilon)) \subset T^{-1}(G)$
 Hence, $T^{-1}(G)$ is open.

$\Leftarrow$ Let $\{x_n\} \subset X, x_n \rightarrow x_0$.

Fix $\varepsilon > 0, T^{-1}(B(Tx_0, \varepsilon))$ is open $\rightarrow \exists \delta > 0$ s.t. $B(x_0, \delta) \subset T^{-1}(B(Tx_0, \varepsilon))$

$\rightarrow T(B(x_0, \delta)) \subset B(Tx_0, \varepsilon)$ $T(B(x_0, \delta)) \subset T(T^{-1}(B(Tx_0, \varepsilon))) \subset B(Tx_0, \varepsilon)$

Because $x_n \rightarrow x_0, \exists N_0 \in \mathbb{N}$ s.t. $d(x_n, x_0) < \delta, \forall n \geq N_0 \rightarrow x_n \in B(x_0, \delta)$
 $\Rightarrow Tx_n \in B(Tx_0, \varepsilon) \rightarrow d(Tx_n, Tx_0) < \varepsilon, \forall n \geq N_0 \rightarrow T$ is continuous.

Corollary: T is continuous $\Leftrightarrow \forall$ closed $C \subset Y, T^{-1}(C)$ is closed in X .

(since $f^{-1}(E^c) = [f^{-1}(E)]^c$) 两个连续的定义

Direct sum 直和: Given two functions $f: X \rightarrow Y$ and $g: X \rightarrow Z$, one can define their direct sum $f \oplus g: X \rightarrow Y \times Z$ defined by $f \oplus g(x) := (f(x), g(x))$.
 Cartesian product.

Lemma Let $f: X \rightarrow \mathbb{R}$ and $g: X \rightarrow \mathbb{R}$, $f \oplus g: X \rightarrow \mathbb{R}^2$. We give $\mathbb{R}^2$ the Euclidean metric.

(a) If $x_0 \in X$, f and g are both continuous at x_0 iff $f \oplus g$ is continuous at x_0 .
 欧几里得度量

(b) f and g are both continuous iff $f \oplus g$ is continuous.

Theorem (a) Let $f_1, \dots, f_k$ be real functions on a metric space X , and let f be the mapping of X into $\mathbb{R}^k$ defined by $f(x) = (f_1(x), \dots, f_k(x))$. ($x \in X$); then f is continuous if and only if each of the functions $f_1, \dots, f_k$ is continuous.

(b) If f and g are continuous mappings of X into $\mathbb{R}^k$, then $f+g$ and $f \cdot g$ are continuous on X .
 解释: 对应分量相乘相加

Lemma Suppose that $g: D \rightarrow \mathbb{R}$ (or $\mathbb{C}$) is continuous at $p \in D$, and that $g(p) \neq 0$.

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连续有通带性(?)

Then there exists $\delta > 0$ with the property that for all $x \in D$ with $|x - p| < \delta$, $g(x) \neq 0$ as well.

pf: Let $\varepsilon = \frac{|g(p)|}{2} > 0$. Because g is continuous at p , $\exists \delta > 0$ s.t. $|x - p| < \delta$, $|g(x) - g(p)| < \varepsilon = \frac{|g(p)|}{2} \rightarrow |g(x)| > \frac{|g(p)|}{2} > 0$ 不等于0等价于绝对值大于0

Theorem Suppose f is a continuous mapping of a compact metric space X into a metric space Y . Then $f(X)$ is compact. **连续, 把紧集映成紧集.**

pf: Let $\{V_\alpha\}$ be an open cover of $f(X)$.

Because f is continuous, each of the sets $f^{-1}(V_\alpha)$ is open

Because X is compact, $X \subset f^{-1}(V_{\alpha_1}) \cup \dots \cup f^{-1}(V_{\alpha_n})$

Because $f(f^{-1}(E)) \subseteq E$ for $E \subset Y$, $f(X) \subset f(f^{-1}(V_{\alpha_1}) \cup \dots \cup f^{-1}(V_{\alpha_n})) \subseteq V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$

Hence, $f(X)$ is compact.

Corollary If f is a continuous mapping of a compact metric space X into $\mathbb{R}^k$, then $f(X)$ is closed and bounded. Then f is bounded. **还可以说**

其实本质还是连续函数把紧集映成紧集, 只不过紧集在 $\mathbb{R}^k$ 中有另一等价命题: 即闭又有界, 从而再推出 f 有界. **one-to-one**

Theorem Suppose f is a continuous mapping of a compact metric space X into a metric space Y . Then the inverse mapping f^{-1} defined on Y by

一个 $f^{-1}(f(x)) = x$ ($x \in X$) is a continuous mapping of Y onto X . **单射, 逆映射存在**

Definition (Uniform continuity - 一致连续) $f: (X, d) \rightarrow (Y, d_1)$ is uniformly continuous if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x, y \in X, d(x, y) < \delta$, then $d_1(f(x), f(y)) < \varepsilon$. (For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(f(x), f(x')) < \varepsilon$ whenever $x, x' \in X$ are such that $d(x, x') < \delta$.)

whenever 后面接前题

$$x = \frac{1}{n}, y = \frac{2}{n}, |x - y| = \frac{1}{n} < \delta$$

$$|f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \frac{|y - x|}{xy}$$

连续: 对于任何给定的点 x 和任何给定的 $\varepsilon > 0$, 存在 $\delta > 0$ 使得当 $|x - x'| < \delta$ 时, 都有 $|f(x) - f(x')| < \varepsilon$.

一致连续: 对于任何给定的 $\varepsilon > 0$, 存在 $\delta > 0$ 使得对于所有的 x 和 y , 只要 $|x - y| < \delta$, 就有 $|f(x) - f(y)| < \varepsilon$.

eg. $\frac{1}{x}$

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 $\delta > 0$

$$|x - x_0| < \delta \Rightarrow \frac{|x_0 - x|}{xx_0} < \varepsilon$$

$$-\delta < x - x_0 < \delta$$

$$x > x_0 - \delta > \frac{2}{3}x_0$$

$$\delta = \frac{x_0}{3}, x^2 < \frac{4}{9}x_0^2$$

例:

i) The function $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous on $(0, 1)$.

continuous: Fix $x_0 \in (0, 1)$. $\forall \varepsilon > 0$. $|f(x) - f(x_0)| = \frac{|x_0 - x|}{xx_0} < \varepsilon$

$$\delta = \min\left\{\frac{x_0}{2}, \frac{x_0^2 \varepsilon}{2}\right\} \Rightarrow \delta \leq \frac{x_0}{2}, \delta \leq \frac{x_0^2 \varepsilon}{2}$$

$$\text{Because } |x - x_0| < \delta \Rightarrow x > x_0 - \delta > \frac{x_0}{2}$$

$$|f(x) - f(x_0)| = \frac{1}{xx_0} |x - x_0| < \frac{2}{x_0} \cdot \frac{x_0^2 \varepsilon}{2} = \varepsilon.$$

not u.c.: Take $\varepsilon = 1$.

$$\text{Take } N \in \mathbb{Z}^+, N \geq 2. \frac{1}{N} < \delta.$$

$$\text{Let } x = \frac{1}{N}, y = \frac{2}{N}. |x - y| = \frac{1}{N} < \delta$$

$$|f(x) - f(y)| = |N - \frac{N}{2}| = \frac{N}{2} \geq 1$$

ii) For any $a > 0$, the function $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, \infty)$

For $\forall \varepsilon > 0$, let $\delta = \varepsilon a^2$. $|x - y| < \varepsilon a^2$, $x, y \geq a$.

$$|f(x) - f(y)| = \frac{|y - x|}{xy} < \frac{|y - x|}{a^2} < \frac{\varepsilon a^2}{a^2} = \varepsilon.$$

$$\frac{1}{x} - \frac{1}{y} = \left| \frac{1}{a} - \frac{1}{a} \right| = 0$$

iii) The function $g(x) = x^2$ is continuous but not uniformly continuous on $(-\infty, \infty)$.

continuous: Fix $x_0 \in (-\infty, \infty)$. $\forall \varepsilon > 0$, $|g(x) - g(x_0)| = |x^2 - x_0^2| = |x - x_0| |x + x_0|$

$$\delta = \min\left(1, \frac{\varepsilon}{1 + 2|x_0|}\right) \Rightarrow \delta \leq 1, \delta \leq \frac{\varepsilon}{1 + 2|x_0|}$$

$$\text{Because } |x - x_0| < 1 \Rightarrow |x + x_0| < |x - x_0| + 2|x_0| < 1 + 2|x_0|$$

$$|g(x) - g(x_0)| = |x - x_0| |x + x_0| < \frac{\varepsilon}{1 + 2|x_0|} \cdot (1 + 2|x_0|) = \varepsilon$$

not u.c.: Take $\varepsilon = 1$. $\delta > 0$

$$\text{Take } x = \frac{1}{\delta}, y = \frac{1}{\delta} + \frac{\delta}{2}, |x - y| = \frac{\delta}{2} < \delta.$$

$$|f(x) - f(y)| = \left| \frac{1}{\delta} - \frac{1}{\frac{1}{\delta} + \frac{\delta}{2}} \right| = \left| \frac{1}{\delta} - \frac{2}{1 + \frac{\delta^2}{2}} \right| = \left| \frac{1 + \frac{\delta^2}{2} - 2}{\delta(1 + \frac{\delta^2}{2})} \right| = \left| \frac{-1 + \frac{\delta^2}{2}}{\delta(1 + \frac{\delta^2}{2})} \right| > \frac{1}{2}$$

Theorem If $f: (X, d_X) \rightarrow (Y, d_Y)$ is continuous and X is compact, then f is uniformly continuous on X .

pf: $\forall \varepsilon > 0$, for $x \in X$.

$\because f$ is continuous at $x \rightarrow \exists \delta_x > 0$ s.t. $d(z, x) < \delta_x, d(f(z), f(x)) < \frac{\varepsilon}{2}$

Let $\bigcup_{x \in X} B(x, \frac{\delta_x}{2}) = X$. $\because X$ is compact $\rightarrow \exists x_1, \dots, x_n \in X$ s.t.

$$\bigcup_{i=1}^n B(x_i, \frac{\delta_{x_i}}{2}) \supset X.$$

Campus

值域集的范围是够多的

使得 x 一定落在某开球内

$$\text{Let } \delta = \min \left\{ \frac{\delta_{x_1}}{2}, \dots, \frac{\delta_{x_n}}{2} \right\}.$$

If $x, y \in X$, $d(x, y) < \delta$. Then, let $x \in B(x_{i_0}, \frac{\delta_{x_{i_0}}}{2})$

$$d(y, x_{i_0}) \leq d(y, x) + d(x, x_{i_0}) < \delta + \frac{\delta_{x_{i_0}}}{2} \leq \frac{\delta_{x_{i_0}}}{2} + \frac{\delta_{x_{i_0}}}{2} = \delta_{x_{i_0}}.$$

$$d(x, x_{i_0}) < \frac{\delta_{x_{i_0}}}{2} < \delta_{x_{i_0}}.$$

由题设得出

$$d(f(y), f(x)) \leq d(f(y), f(x_{i_0})) + d(f(x_{i_0}), f(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

f is U.C..

右极限

Definition $f: (a, b) \rightarrow \mathbb{R}$, $x \in (a, b)$. We say $f(x+) = q$ if $\forall t_n \in (x, b)$ with $t_n \rightarrow x$, $f(t_n) \rightarrow q$. $f(x-) = q$ if $\forall t_n \in (a, x)$ with $t_n \rightarrow x$, $f(t_n) \rightarrow q$.

Definition Let f be defined on (a, b) . If f is discontinuous at a point x , and if $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the first kind, or a simple discontinuity, at x . Otherwise, ... second kind.

eg. (a) $f(x) = \begin{cases} x & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$ has a discontinuity of the second kind at every point x , since neither $f(x+)$, $f(x-)$ do not exist.

$p f: x \in \mathbb{R}, \exists t_n \in \mathbb{Q}, t_n > x_0, t_n \rightarrow x_0, f(x_0+) = 1$

$f(x-)$ 同理

$$\exists x_n \in \mathbb{Q}^c, x_n > x_0, x_n \rightarrow x_0, f(x_0+) = 0.$$

(b) $f(x) = \begin{cases} x & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$ is continuous at $x=0$ and has a discontinuity of the second kind at every other point.

$$0 \leq |f(x) - f(0)| \leq |x| \xrightarrow{x \rightarrow 0} 0 \text{ 这个夹逼定理用的好哇}$$

$$x_0 \neq 0, \exists t_n \in \mathbb{Q}, t_n > x_0, t_n \rightarrow x_0, f(x_0+) = x_0$$

$$\exists x_n \in \mathbb{Q}^c, x_n > x_0, x_n \rightarrow x_0, f(x_0+) = 0$$

故 $f(x+)$, $f(x-)$ 不存在. 第2类不连续.

(c) Define $f(x) = \begin{cases} \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$. Since neither $f(0+)$ nor $f(0-)$ exists, f has a discontinuity of the second kind at $x=0$.

$$\text{If } x_n = \frac{1}{2n\pi + \frac{\pi}{2}}, n \rightarrow +\infty, x_n \rightarrow 0, f(0+) = \lim_{n \rightarrow \infty} (2n\pi + \frac{\pi}{2}) = 1$$

$$\text{If } x_n = \frac{1}{2n\pi}, n \rightarrow +\infty, x_n \rightarrow 0, f(0+) = \lim_{n \rightarrow \infty} (2n\pi) = 0.$$

重在找到如何趋近

故 $f(0+)$ 不存在. $f(0-)$ 同理

单侧函数, 无穷极限与在^{无穷处}极限的极限, 实值函数的导数.

Monotonic functions, infinite limits and limit at infinity, the derivative of a real function

如何求左, 右极限

与数列的方法有关

Lemma Let f be defined on (a, b) and consider any $x \in (a, b]$. $f(x-) = A$

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ with $a < x - \delta$, such that if $x - \delta < t < x$ then $|f(t) - A| < \varepsilon$.

pf: $\Leftarrow$ Let $t_n \in (a, x)$, $t_n \rightarrow x$. We Need To Show $f(t_n) \rightarrow A$ (i.e. $\forall \varepsilon > 0, \exists N$ s.t. $|f(t_n) - A| < \varepsilon$ for $n \geq N$).

Because $x - \delta < t < x$ then $|f(t) - A| < \varepsilon$, $t_n \rightarrow x$.

Then for $\delta > 0$, $\exists N$ s.t. $d(t_n, x) < \delta \rightarrow x - \delta < t_n < x \rightarrow |f(t_n) - A| < \varepsilon$.

(逆否?) Hence $f(t_n) \rightarrow A$, $n \rightarrow \infty$

for $n \geq N$

$\Rightarrow$ If the conclusion were false, then $\exists \varepsilon > 0$ s.t. for any $\delta > 0$ with $a < x - \delta$, $\exists t \in (x - \delta, x)$ s.t. $|f(t) - A| \geq \varepsilon$.

Let $\delta_n = \frac{1}{n}$, $t_n \in (x - \frac{1}{n}, x)$ when n is large enough $\rightarrow |f(t_n) - A| \geq \varepsilon$

So $t_n \in (a, x)$, $t_n \rightarrow x$, but $\lim_{n \rightarrow \infty} f(t_n) \neq A$.

Contradicts.

Theorem Let f be monotonically increasing on (a, b) . Then $f(x+)$ and $f(x-)$ exist at every point of x of (a, b) . More precisely

$$(1) \quad \sup_{a < t < x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x < t < b} f(t)$$

Furthermore, if $a < x < y < b$, then $f(x+) \leq f(y-)$.

pf: 因为单增, the set of numbers $f(t)$, where $a < t < x$, is bounded above by $f(x)$

令 A 为上确界. 则 $A \leq f(x)$. 接下来证: $A = f(x-)$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $a < x - \delta < x$, $A - \varepsilon < f(x - \delta) \leq A$.

因为 f 单增, $f(x - \delta) \leq f(t) \leq A$ ($x - \delta < t < x$)

Because $A - \varepsilon < f(t) \leq A$, $|f(t) - A| < \varepsilon \rightarrow f(x-) = A$

因为单增, the set of numbers $f(t)$, where $x < t < b$, is bounded below by $f(x)$

Let B be the least lower bound. Let $B \geq f(x)$. Then prove: $B = f(x+)$.

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $x < x + \delta < b$, $B + \varepsilon > f(x + \delta) \geq B$.

因为 f 单调, $f(x+\delta) \geq f(t) \geq B$ ($x < t < x+\delta$)

Because $B+\varepsilon > f(t) \geq B$. $|f(t)-B| < \varepsilon \Rightarrow f(x+)=B$.

Next.

If $a < x < y < b$. $f(x+) = \inf_{x < t < b} f(t) = \inf_{x < t < y} f(t)$, $f(y-) = \sup_{a < t < y} f(t) = \sup_{x < t < y} f(t)$

Hence, $f(x+) \leq f(y-)$.

Corollary Monotonic functions have no discontinuities of the second kind.

单调函数没有第二类不连续点 $f(x-)$ 和 $f(x+)$ 都存在

Theorem Let f be monotonic on (a,b) . Then the set of points of (a,b) at which f is discontinuous is at most countable. 单调函数的不连续点集为至多可数集!

p.p: Let E be the set of points at which f is discontinuous. $x \in E$ 有限或可列

Because $f(x-) < f(x) < f(x+)$. Let $x_1 < x_2$, $f(x_1+) \leq f(x_2-) \rightarrow r(x_1) \neq r(x_2)$

We establish a 1-1 correspondence between the set E and a subset of the set of rational numbers. 若 $x_1 \neq x_2$

neighbourhood of $+\infty: (c, +\infty)$, neighbourhood of $-\infty: (-\infty, c)$.

infinite adherent points. 无穷粘滞点: $+\infty$ is adherent to X iff X has no upper bound, $\inf(X) = -\infty$ or equivalently iff $\sup(X) = +\infty$. A set is bounded iff and only iff $+\infty$ and $-\infty$ are not adherent points.

Definitions

1. We say that $f(x)$ approaches infinity as x approaches c , and write $\lim_{x \rightarrow c} f(x) = \infty$,

if for every positive real number B there exists a corresponding $\delta > 0$ such that

$f(x) > B$ whenever $0 < |x-c| < \delta$.

2. We say that $f(x)$ approaches negative infinity as x approaches c , and write $\lim_{x \rightarrow c} f(x) = -\infty$,

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ s.t.

$f(x) < -B$ whenever $0 < |x-c| < \delta$

Definition Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient $\phi(t) = \frac{f(t) - f(x)}{t - x}$ ($a < t < b, t \neq x$), and define **differentiable** $f'(x) = \lim_{t \rightarrow x} \phi(t)$, provided this limit exists. $\perp$

Remark 1 Assume $f'(x)$ exists, let $\frac{f(t) - f(x)}{t - x} - f'(x) = u(t)$, then

$$f(t) = f(x) + (t - x)(f'(x) + u(t)), \quad \lim_{t \rightarrow x} u(t) = 0$$

Remark 2 Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x . **可导一定连续**

(pf: As $t \rightarrow x$, $f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \rightarrow f'(x) \cdot 0 = 0$)

Remark 3 $f'(x)$ exists, $f'(x) = A \Leftrightarrow \lim_{t \rightarrow x} \left| \frac{f(t) - f(x)}{t - x} - A \right| = 0 \Leftrightarrow$

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that if } 0 < |t - x| < \delta, \text{ then } \left| \frac{f(t) - f(x)}{t - x} - A \right| \leq \varepsilon$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. if } |t - x| < \delta, \text{ then } |f(t) - f(x) - A(t - x)| \leq \varepsilon |t - x|$$

Remark 4 $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$ if f and g are differentiable at a .

(pf: Let $h = f/g$.

Because $f(t) \rightarrow f(x)$ as $t \rightarrow x$, $g(t) \rightarrow g(x)$ as $t \rightarrow x$

$$\frac{h(t) - h(x)}{t - x} = \frac{1}{g(t)g(x)} \left[g(x) \frac{f(t) - f(x)}{t - x} - f(x) \frac{g(t) - g(x)}{t - x} \right]$$

Let $t \rightarrow x$, done.

Theorem Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f , and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$, ($a \leq t \leq b$).

Then h is differentiable at x , and $h'(x) = g'(f(x))f'(x)$.

(pf: Let $y = f(x)$. Because the derivative

$$f(t) - f(x) = (t - x)[f'(x) + u(t)], \quad g(s) - g(y) = (s - y)[g'(y) + v(s)], \quad t \in [a, b], \quad s \in I$$

$$u(t) \rightarrow 0 \text{ as } t \rightarrow x, \quad v(s) \rightarrow 0 \text{ as } s \rightarrow y. \text{ Let } s = f(t)$$

$$h(t) - h(x) = g(f(t)) - g(f(x)) = [f(t) - f(x)] \cdot [g'(y) + v(s)]$$

$$= (t - x) \cdot [f'(x) + u(t)] \cdot [g'(y) + v(s)]$$

$$\text{If } t \neq x, \quad \frac{h(t) - h(x)}{t - x} = [g'(y) + v(s)] \cdot [f'(x) + u(t)] \rightarrow g'(y) \cdot f'(x) = g'(f(x)) \cdot f'(x)$$

Definition (Local maxima and minima) Let f be a real function defined on a metric space X . We say that f has a local maximum at a point $p \in X$ if there exists $\delta > 0$ s.t. $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$. Then, f has a local minimum at a point $p \in X$ if there exists $\delta > 0$ s.t. $f(q) \geq f(p)$ for all $q \in X$ with $d(p, q) < \delta$. **重点是找到一个区间**

是个值 ($\mathbb{R}$)

eg. $f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2 - x^4$ has local minimum at 0 .

Because when we choose $\delta = 1$ and restrict f to the interval $(-1, 1)$.

For $x \in (-1, 1)$ we have $x^4 \leq x^2$ thus $f(x) = x^2 - x^4 \geq 0 = f(0)$.

Hence, $f|_{(-1, 1)}$ has a local minimum at 0 .

Theorem Let f be defined on $[a, b]$, if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$. **①一定要是开区间, 因为闭区间的端点不好讨论** **②反推不成立. eg. $f(x) = x^3, x > 0$.**

(pf: Let $\delta > 0, a < x - \delta < x < x + \delta < b$

If $x - \delta < t < x$, then $\frac{f(t) - f(x)}{t - x} \geq 0$. Letting $t \rightarrow x$, $f'(x) \geq 0$.

If $x < t < x + \delta$, then $\frac{f(t) - f(x)}{t - x} \leq 0$. Letting $t \rightarrow x$, $f'(x) \leq 0$.

Hence, $f'(x) = 0$.

Theorem $f: [a, b] \rightarrow \mathbb{R}$ is continuous $f(a) < 0 < f(b) \Rightarrow \exists \delta \in (a, b)$ s.t. $f(\delta) = 0$.

(pf: Let $c_1 = \frac{1}{2}(a+b)$. if $c_1 = 0$, done. 也可以用 $\{f(a_n)\}, \{f(b_n)\}$ 都收敛到 0 来做
Otherwise, i) if $f(c_1) < 0$, let $a_2 = c_1, b_2 = b, f(a_2) < 0, f(b_2) > 0$.

ii) if $f(c_1) > 0$, let $a_2 = a, b_2 = c_1, f(a_2) < 0, f(b_2) > 0$.

Continuing this process, we will obtain a point C_n with $f(C_n) = 0$

or $[a_1, b_1] \supset [a_2, b_2] \supset \dots [a_n, b_n] \supset \dots$ with $f(a_n) < 0, f(b_n) > 0$

$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \dots = \frac{1}{2^{n-1}}(b_1 - a_1) \rightarrow 0$. **还要加上数列收敛的条件**

Corollary If f is continuous, $f(a) \leq k \leq f(b)$, there exists some $x_0 \in [a, b]$ with $f(x_0) = k$.

(pf: Define $g: [a, b] \rightarrow \mathbb{R}, x \mapsto f(x) - k$. **能取到中间值的前提是连续**

$g(a) = f(a) - k \leq 0, g(b) = f(b) - k \geq 0$. **端点可以单独讨论**

By before theorem, $\exists x_0 \in [a, b]$ s.t. $g(x_0) = 0$ i.e. $f(x_0) = k$.) KOKUYO

罗尔定理 前提: 闭区间上连续, 开区间上可导

Theorem (Rolle's theorem) Suppose f is a real-valued function, which is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) .

Suppose that $f(a) = f(b)$. Then there exists $c \in (a, b)$ s.t. $f'(c) = 0$.

(pf: If $f(x) = f(a)$ for all $x \in (a, b)$ then $f'(x) \equiv 0$ for $x \in (a, b)$.

Otherwise f is not monotonic on $[a, b]$, and therefore has a local max.

or loc. min. at an interior point c of $[a, b]$. Then $f'(c) = 0$.)

Proposition If f is a continuous function on an interval I then f is injective if and only if f is strictly monotonic. 单射

Corollary Suppose that $f'(x) \neq 0$ for each $x \in (a, b)$. Then f is strictly monotonic.

(pf: If not, f is not injective and so there $\exists a' < b' \leq b$ s.t. $f(a') = f(b')$.

Then there $\exists c, a' < c < b'$ with $f'(c) = 0$. Contradicts.)

结合单射与单调关系和罗尔定理

(pf: Let $g(t) = f(t) - \lambda t$, because $g(t)$ is continuous, it has minimum in $[a, b]$.

Because $g'(a) < 0$, $\exists t_1 \in (a, b)$ s.t. $g(t_1) < g(a)$.

接下来要做的就是
证其落在开区间内

$g'(b) > 0$, $\exists t_2 \in (a, b)$ s.t. $g(t_2) < g(b)$.

Hence, g attains its minimum on $[a, b]$ at $\lambda \in (a, b)$.

i.e. $g'(\lambda) = 0$ and so $f'(\lambda) = \lambda$

Theorem Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t-\alpha)^k$.

Then there exists a point λ between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(\lambda)}{n!} (\beta-\alpha)^n = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta-\alpha)^k + \frac{f^{(n)}(\lambda)}{n!} (\beta-\alpha)^n.$$

(pf: Define M by $f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta-\alpha)^k + M(\beta-\alpha)^n$ ($f^{(0)}(\alpha) = f(\alpha)$).

We will show that $\exists \lambda \in (a, b)$ with $M = \frac{f^{(n)}(\lambda)}{n!}$

Define $g: [a, b] \rightarrow \mathbb{R}$, $t \mapsto g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t-\alpha)^k - M(t-\alpha)^n$

$$\Rightarrow g^{(n)}(t) = f^{(n)}(t) - n!M$$

Because $P^{(n)}(\alpha) = f^{(n)}(\alpha)$, thus $g(\alpha) = g'(\alpha) = \dots = g^{(n-1)}(\alpha) = 0$

Because $g(\beta) = 0$, $\exists \lambda_1 \in (\alpha, \beta)$ s.t. $g'(\lambda_1) = 0$

Because $g'(\alpha) = 0$, $\exists \lambda_2 \in (\alpha, \lambda_1)$ s.t. $g''(\lambda_2) = 0$

Because $g''(\alpha) = 0$, $\exists \lambda_3 \in (\alpha, \lambda_2)$ s.t. $g'''(\lambda_3) = 0$

i.e. $f^{(n)}(\lambda) = n!M \Rightarrow M = \frac{f^{(n)}(\lambda)}{n!}$

Definition $f: I \rightarrow \mathbb{R}$ is convex if $E = \{(x, y) : x \in I, y \geq f(x)\}$ is convex.

$$\Leftrightarrow f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2), \lambda \in [0, 1] \quad \text{concave up}$$

$$f \text{ is strictly convex} \Leftrightarrow f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda)f(x_2), \lambda \in [0, 1]$$

$$f \text{ is concave} \Leftrightarrow f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2), \lambda \in [0, 1] \quad \text{concave down}$$

$$f \text{ is strictly concave} \Leftrightarrow f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda)f(x_2), \lambda \in [0, 1]$$

Prop. ① If f and g are convex functions on an interval I and $a > 0$ then $f+g$ and af are convex.

证: 直接代入不等式

② If $(f_n)_{n \in \mathbb{N}}$ is a sequence of convex functions on an interval I , and if $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$, for each $x \in I$, then f is convex.

$$\text{证: } \forall x, y \in I, t \in [0, 1], f((1-t)x + ty) = \lim_{n \rightarrow \infty} f_n((1-t)x + ty) \leq \lim_{n \rightarrow \infty} [(1-t)f_n(x) + tf_n(y)] = (1-t)f(x) + tf(y)$$

③ If $\{f: f \in F\}$ is a family of convex functions on an interval I for which $g(x) = \sup_{f \in F} f(x)$ is finite for each $x \in I$ then g is convex.

$$\text{证: } \forall \varepsilon > 0, \exists f(x_0) \geq g(x_0) - \varepsilon.$$

$$\begin{aligned} g(x_0) - \varepsilon &\leq f(x_0) \leq (1-t)f(x_0) + tf(x_1) \leq (1-t)g(x_0) + tg(x_1) \\ &\Rightarrow g(x_0) \leq (1-t)g(x_0) + tg(x_1) \end{aligned}$$

④ If f, g are convex, non-negative increasing functions on interval I , then fg is convex.

$$\text{Because } f \text{ and } g \text{ are increasing, } (g(x_1) - g(x_0))(f(x_1) - f(x_0)) \geq 0$$

$$\Rightarrow f(x_0)g(x_1) + f(x_1)g(x_0) \leq f(x_0)g(x_0) + f(x_1)g(x_1)$$

$$f(x_0)g(x_0) \leq ((1-t)f(x_0) + tf(x_1))((1-t)g(x_0) + tg(x_1))$$

$$= (1-t)^2 f(x_0)g(x_0) + t(1-t)(f(x_0)g(x_1) + f(x_1)g(x_0)) + t^2(f(x_1)g(x_1))$$

$$\leq (1-t)^2 f(x_0)g(x_0) + t(1-t)(f(x_0)g(x_0) + f(x_1)g(x_1)) + t^2(f(x_1)g(x_1))$$

$$= (1-t)f(x_0)g(x_0) + tf(x_1)g(x_1)$$

⑤ If f is a convex function on an interval I , and if ϕ is an increasing convex function on an interval J which contains $f(I)$, then $\phi \circ f$ is a convex function on I . 外函数单增且凸, 内函数凸, 则复合函数凸

$$\phi(f(x)) \leq \phi((1-t)f(x_1) + tf(x_2)) \leq (1-t)\phi(f(x_1)) + t\phi(f(x_2))$$

Theorem $f: (a, b) \rightarrow \mathbb{R}$ that is differentiable on the open interval (a, b) to be convex.

$\Leftrightarrow f'$ is nondecreasing on (a, b) $\Leftrightarrow f''(x) \geq 0$. 不取等就是严格

pf: $\Rightarrow$ Let $x_1, x_2 \in (a, b)$, $x_1 < x_2$ and let $x \in (x_1, x_2)$

We have $x = tx_1 + (1-t)x_2$, where $t = \frac{x_2 - x}{x_2 - x_1}$.

Because f is convex, $f(x) \leq \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2) \Rightarrow \frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$. (*)

Letting x in (*) tend first to x_1 , then to x_2 , we get $f'(x_1) \leq f'(x_2)$

$\Leftarrow$ For $a < \alpha < x < x_2 < b$, we have $\frac{f(x) - f(x_1)}{x - x_1} = f'(\alpha)$, $\frac{f(x_2) - f(x)}{x_2 - x} = f'(\beta)$, for $x_1 < \alpha < x < \beta < x_2$.

If $f'(\alpha) \leq f'(\beta)$, we get (*) again.

Jensen's inequality: If $f: (a, b) \rightarrow \mathbb{R}$ is a convex function, $x_1, \dots, x_n$ are points of (a, b) , and $\alpha_1, \dots, \alpha_n$ are nonnegative numbers such that $\alpha_1 + \dots + \alpha_n = 1$, then

$$f(\alpha_1 x_1 + \dots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \dots + \alpha_n f(x_n)$$

pf: Assume $\alpha_n \neq 0$, the $\beta = \alpha_1 + \dots + \alpha_n > 0$, $\frac{\alpha_1}{\beta} + \dots + \frac{\alpha_n}{\beta} = 1$

Because f is convex, $f(\alpha_1 x_1 + \dots + \alpha_n x_n) = f(\alpha_1 x_1 + \beta(\frac{\alpha_2}{\beta} x_2 + \dots + \frac{\alpha_n}{\beta} x_n)) \leq \alpha_1 f(x_1) + \beta f(\frac{\alpha_2}{\beta} x_2 + \dots + \frac{\alpha_n}{\beta} x_n)$. (Because $\alpha_1 + \beta = 1$, $(\frac{\alpha_2}{\beta} x_2 + \dots + \frac{\alpha_n}{\beta} x_n) \in (a, b)$).

By induction hypothesis, we have $f(\frac{\alpha_2}{\beta} x_2 + \dots + \frac{\alpha_n}{\beta} x_n) \leq \frac{\alpha_2}{\beta} f(x_2) + \dots + \frac{\alpha_n}{\beta} f(x_n)$.

Hence, $f(\alpha_1 x_1 + \dots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \dots + \alpha_n f(x_n)$.

用凸性证明不等式

$\frac{1}{x}$ $\frac{1}{x}$

eg. 1. Because $f(x) = -\ln x$ is strictly convex, $x \in (0, +\infty)$. For $\sum_{i=1}^n \alpha_i = 1$

$$-\ln(\alpha_1 x_1 + \dots + \alpha_n x_n) \leq -\alpha_1 \ln x_1 - \dots - \alpha_n \ln x_n$$

for $x_i > 0, \alpha_i \geq 0, i = 1, \dots, n$

$$\ln(\alpha_1 x_1 + \dots + \alpha_n x_n) \geq \alpha_1 \ln x_1 + \dots + \alpha_n \ln x_n \text{ i.e. } \alpha_1 x_1 + \dots + \alpha_n x_n \geq x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

In particular, if $\alpha_1 = \dots = \alpha_n = \frac{1}{n}$, we obtain the classical inequality.

$$\sqrt[n]{x_1 \dots x_n} \leq \frac{x_1 + \dots + x_n}{n}$$

在多维上, 向量函数

Theorem Suppose f is a continuous mapping of $[a, b]$ into $\mathbb{R}^k$ and f is differentiable in (a, b) . Then there exists $\xi \in (a, b)$ s.t. $|f(b) - f(a)| \leq (b-a) |f'(\xi)|$

Definition $f: I \rightarrow \mathbb{R}$, $x \in I$. If $\lim_{t \rightarrow x^-} \frac{f(t) - f(x)}{t - x}$ exists, we say f has left derivative at x . $f'(x-) = \lim_{t \rightarrow x^-} \frac{f(t) - f(x)}{t - x}$. If $\lim_{t \rightarrow x^+} \frac{f(t) - f(x)}{t - x}$ exists, we say f has right derivative at x . $f'(x+) = \lim_{t \rightarrow x^+} \frac{f(t) - f(x)}{t - x}$. $f'(x-)$ exists, then $f(x)$ is left continuous at x . $f'(x+)$ exists, then $f(x)$ is right continuous at x . f is differentiable at an interior point a iff it is differentiable on the right and on the left and $f'(a+) = f'(a-)$. $f(x)$ 左连续 + $f(x)$ 右连续 $\rightarrow f(x)$ 连续

Proposition Suppose that f is a real-valued function on an open interval I . The following are equivalent.

(i) f is convex.

$a \quad b \quad c$

(ii) If $a, b, c \in I$ and $a < b < c$ then $\frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(a)}{c - a}$.

(iii) If $a, b, c \in I$ and $a < b < c$ then $\frac{f(c) - f(a)}{c - a} \leq \frac{f(c) - f(b)}{c - b}$.

(iv) If $a, b, c \in I$ and $a < b < c$ then $\frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(b)}{c - b}$.

pf of (ii): Let $t = \frac{b-a}{c-a}$, $0 < t < 1$, $1-t = \frac{c-b}{c-a}$, and $b = \frac{c-b}{c-a}a + \frac{b-a}{c-a}c = (1-t)a + tc$

Because f is convex, $f(b) \leq \frac{c-b}{c-a}f(a) + \frac{b-a}{c-a}f(c)$

$f(b) - f(a) \leq \frac{(c-b) - (c-a)}{c-a}f(a) + \frac{b-a}{c-a}f(c) = \frac{b-a}{c-a}(f(c) - f(a))$

$\rightarrow \frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(a)}{c - a}$

$b = \frac{b-a}{c-a}c + \frac{c-b}{c-a}a \rightarrow f(b) \leq \frac{b-a}{c-a}f(c) + \frac{c-b}{c-a}f(a) \rightarrow f(c) - f(a) \leq \frac{c-a}{c-b}(f(c) - f(b))$

$\rightarrow f(b) \leq \frac{c-b}{c-a}f(a) + \frac{b-a}{c-a}f(c) \rightarrow f(b) - f(a) \leq \frac{b-a}{c-b}(f(c) - f(b))$

Theorem Suppose that f is a convex function on an open interval I . 入的变换被取到

(i) f is continuous on I . 凸函数一定连续, 因为区间中的每个点都有可能因为

(ii) f is differentiable on the right and on the left at each point a of I , and $f'(a-) \leq f'(a+)$.

(iii) If $a < b$ then $f'(a+) \leq f'(b-)$.

- (iv) The mapping $a \rightarrow f'(a+)$ is increasing, and is continuous on the right at each point a of I . 虽然导数不一定处处存在, 但有增大的趋势
- (v) The mapping $a \rightarrow f'(a-)$ is increasing, and is continuous on the left at each point a of I .

The Riemann-Stieltjes Integral 黎曼-斯蒂尔杰斯积分

Definition Let $[a, b]$ be a given interval. By a partition P of $[a, b]$ we mean a finite set of points $x_0, x_1, \dots, x_n$, where $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$. Then $\Delta x_i = x_i - x_{i-1}$ ($i = 1, \dots, n$)

Now suppose f is a bounded real function defined on $[a, b]$. Corresponding to each partition P of $[a, b]$ we put

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x), \quad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x)$$

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i \quad \text{upper Riemann integral}, \quad L(P, f) = \sum_{i=1}^n m_i \Delta x_i \quad \text{lower Riemann integral}$$

$$\int_a^b f dx = \inf U(P, f) \quad \text{Riemann integral}, \quad \int_a^b f dx = \sup L(P, f)$$

where the $\inf$ and $\sup$ are taken over all partitions P of $[a, b]$.

If the upper and lower integrals are equal, we say that f is Riemann-integrable on $[a, b]$. we write $f \in \mathcal{R}$ ($\mathcal{R}$ denotes the set of Riemann-integrable functions), and we denote the common value of (1) and (2) by $\int_a^b f dx$.

Definition Let α be a monotonically increasing function on $[a, b]$. Corresponding to each partition P of $[a, b]$, we write $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. ($\Delta \alpha_i > 0$).

For any real function f which is bounded on $[a, b]$ we put

$$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta \alpha_i, \quad L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta \alpha_i$$

$$\int_a^b f d\alpha = \inf U(P, f, \alpha) \quad (*), \quad \int_a^b f d\alpha = \sup L(P, f, \alpha) \quad (**)$$

the $\inf$ and $\sup$ again being taken over all partitions

If $(*)$ and $(**)$ are equal, we denote their common value by $\int_a^b f d\alpha$ or $\int_a^b f d\alpha(x)$.

This is the Riemann-Stieltjes integral (or Stieltjes integral) of f with respect to α , over $[a, b]$. $f \in \mathcal{R}(\alpha)$. $R-S$ Integral $\alpha(x)=x \rightarrow R$ integral

Definition We say that the partition P^* is a refinement of P if $P^* \supset P$ (every point of P is a point of P^*). Given two partitions P_1 and P_2 , P^* is their common refinement if $P^* = P_1 \cup P_2$. P^* 中的端点为 P_1 和 P_2 的端点

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Theorem If P^* is a refinement of P , then $L(P, f, \alpha) \leq L(P^*, f, \alpha)$ and $U(P^*, f, \alpha) \leq U(P, f, \alpha)$ 越细分越接近

pf: Suppose P^* contains just one point more than P , x^* , and suppose $x_{i-1} < x^* < x_i$ where x_{i-1} and x_i are two consecutive points of P .

$$\text{Let } w_1 = \inf f(x) \cdot (x_{i-1} \leq x \leq x^*), \quad w_2 = \inf f(x) \cdot (x^* \leq x \leq x_i)$$

$$m_i = \inf f(x) \cdot (x_{i-1} \leq x \leq x_i)$$

$$\rightarrow w_1 \geq m_i, \quad w_2 \geq m_i$$

$$\text{Hence, } L(P^*, f, \alpha) - L(P, f, \alpha)$$

$$= w_1 [\alpha(x^*) - \alpha(x_{i-1})] + w_2 [\alpha(x_i) - \alpha(x^*)] - m_i [\alpha(x_i) - \alpha(x_{i-1})]$$

$$= (w_1 - m_i) [\alpha(x^*) - \alpha(x_{i-1})] + (w_2 - m_i) [\alpha(x_i) - \alpha(x^*)] \geq 0.$$

If P^* contains k points more than P , we repeat this reasoning k times.

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Theorem $\int_a^b f da \leq \bar{\int}_a^b f da$. 取上和取下的区间分割方式可能不同

pf: Let P^* be the common refinement of two partitions P_1 and P_2 .

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha) \rightarrow L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

First, fix P_2 and take sup over all P_1 , $\int f da \leq U(P_2, f, \alpha)$.

Then, take inf over all P_2 , $\int_a^b f da \leq \bar{\int}_a^b f da$.

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Theorem $f \in \mathcal{R}(\alpha)$ on $[a, b]$ iff for $\forall \varepsilon > 0$, $\exists$ a partition P s.t. $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$. (*)

pf: For every P we have $L(P, f, \alpha) \leq \int f da \leq \bar{\int} f da \leq U(P, f, \alpha)$

$$(*) \rightarrow 0 \leq \bar{\int} f da - \int f da < \varepsilon$$

Because it satisfied for every $\varepsilon > 0$, $\bar{\int} f da = \int f da$. i.e. $f \in \mathcal{R}(\alpha)$

$\Rightarrow$ Because $f \in \mathcal{R}(\alpha)$, $\int f da = \inf U(P, f, \alpha) = \sup L(P, f, \alpha)$

$$\forall \varepsilon > 0, \exists \text{ partitions } P_1 \text{ and } P_2 \text{ s.t. } U(P_2, f, \alpha) - \int f da < \frac{\varepsilon}{2}, \int f da - L(P_1, f, \alpha) < \frac{\varepsilon}{2}$$

Then choose P to be the common refinement of P_1 and P_2 .

No.

Date

A C D X X

A B V X

$$U(P, f, \alpha) \leq U(P, f, \alpha) < \int f d\alpha + \frac{\epsilon}{2} < L(P, f, \alpha) + \epsilon \leq L(P, f, \alpha) + \epsilon.$$

6.7 Theorem (a) If $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ holds for some P and some ϵ , then

this holds (with the same ϵ) for every refinement of P .

(b) If (13) holds for $P = \{x_0, \dots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then $\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta x_i < \epsilon$. 每一段取两个点

(c) If $f \in \mathcal{R}(\alpha)$ and the hypotheses of (b) hold, then $|\sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f d\alpha| < \epsilon$

pf: (a) $U(P^*, f, \alpha) - L(P^*, f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

(b) Because $f(s_i)$ and $f(t_i)$ lie in $[m_i, M_i]$, then $|f(s_i) - f(t_i)| \leq M_i - m_i$

$$\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta x_i \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

(c) Because $L(P, f, \alpha) \leq \sum f(t_i) \Delta x_i \leq U(P, f, \alpha)$, $L(P, f, \alpha) \leq \int f d\alpha \leq U(P, f, \alpha)$,

$$\text{then } |\sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f d\alpha| < \epsilon$$

6.8

连续, 可积

Theorem If f is continuous on $[a, b]$ then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

pf: $\forall \epsilon > 0$, we could find $\eta > 0$ s.t. $[\alpha(b) - \alpha(a)]\eta < \epsilon$.

Because f is u.c. on $[a, b]$, for $x \in [a, b]$, $t \in [a, b]$, $\exists \delta > 0$ s.t. $|x - t| < \delta$ then $|f(x) - f(t)| < \eta$. 在 $\mathbb{R}$ 上, 故一致连续

If P is any partition of $[a, b]$ s.t. $\Delta x_i < \delta$ for all i , then $M_i - m_i \leq \eta$ ($i = 1, \dots, n$)

$$\text{Then } U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^n (M_i - m_i) \Delta x_i \leq \eta \sum_{i=1}^n \Delta x_i = \eta [\alpha(b) - \alpha(a)] < \epsilon.$$

Hence, $f \in \mathcal{R}(\alpha)$.

6.9

单调, 连续, 可积

Theorem If f is monotonic on $[a, b]$, and if α is continuous on $[a, b]$, then $f \in \mathcal{R}(\alpha)$.

pf: For any $\epsilon > 0$.

First, we get choose a partition P s.t. $\Delta x_i = \frac{\alpha(b) - \alpha(a)}{n}$ ($i = 1, \dots, n$). (B.c. α is continuous, it is possible).

Suppose f is monotonically increasing, $M_i = f(x_i)$, $m_i = f(x_{i-1})$ ($i = 1, \dots, n$).

$$\text{So that } U(P, f, \alpha) - L(P, f, \alpha) = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \\ = \frac{\alpha(b) - \alpha(a)}{n} \cdot [f(b) - f(a)] < \varepsilon$$

if n is taken large enough.

6.10 有有限个不连续点 + 有界 + α 在不连续点处连续
Theorem Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous.

Then $f \in \mathcal{R}(\alpha)$. 构造 partition (ppt)

6.11
Theorem Suppose $f \in \mathcal{R}(\alpha)$ on $[a, b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathcal{R}(\alpha)$ on $[a, b]$.
内函数可积, 外函数连续, 复合函数可积

pf: $\forall \varepsilon > 0$. Because ϕ is u.c. on $[m, M]$. For $s, t \in [m, M]$, $\exists \delta > 0$ s.t. $\delta < \varepsilon$, $|s - t| < \delta$, then $|\phi(s) - \phi(t)| < \varepsilon$.

Because $f \in \mathcal{R}(\alpha)$, $\exists$ a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ s.t. $U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$.

Let M_i, m_i for f , M_i^*, m_i^* for h .

Divide the numbers $1, \dots, n$ into two classes: $i \in A$ if $M_i - m_i < \delta$, $i \in B$ if $M_i - m_i \geq \delta$.

For $i \in A$, because $M_i - m_i < \delta$, $M_i^* - m_i^* < \varepsilon$.

For $i \in B$, $M_i^* - m_i^* \leq 2K$, where $K = \sup |\phi(t)|$, $m \leq t \leq M$.

$$\delta \sum_{i \in B} \Delta \alpha_i \leq \sum_{i \in B} (M_i - m_i) \Delta \alpha_i < \delta^2 \rightarrow \sum_{i \in B} \Delta \alpha_i < \delta.$$

$$U(P, h, \alpha) - L(P, h, \alpha) = \sum_{i \in A} (M_i^* - m_i^*) \Delta \alpha_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta \alpha_i$$

$$< \varepsilon [\alpha(b) - \alpha(a)] + 2K \delta$$

$$< \varepsilon [\alpha(b) - \alpha(a) + 2K]$$

6.12

Theorem (a) If $f_1 \in \mathcal{R}(a)$ and $f_2 \in \mathcal{R}(a)$ on $[a, b]$, then $f_1 + f_2 \in \mathcal{R}(a)$, $cf \in \mathcal{R}(a)$ for every constant c , and $\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$, $\int_a^b cf d\alpha = c \int_a^b f d\alpha$.

(b) If $f_1(x) \leq f_2(x)$ on $[a, b]$, then $\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$.

(c) If $f \in \mathcal{R}(a)$ on $[a, b]$ and if $a < c < b$, then $f \in \mathcal{R}(a)$ on $[a, c]$ and on $[c, b]$, and $\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$.

(d) If $f \in \mathcal{R}(a)$ on $[a, b]$ and if $|f(x)| \leq M$ on $[a, b]$, then $|\int_a^b f d\alpha| \leq M[\alpha(b) - \alpha(a)]$.

(e) If $f \in \mathcal{R}(a_1)$ and $f \in \mathcal{R}(a_2)$, then $f \in \mathcal{R}(a_1 + a_2)$ and

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2;$$

if $f \in \mathcal{R}(a)$ and c is a positive constant, then $f \in \mathcal{R}(ca)$ and

$$\int_a^b f d(ca) = c \int_a^b f d\alpha.$$

pf of (a): $\forall \epsilon > 0$. $\therefore f_1, f_2 \in \mathcal{R}(a)$. $\exists P_1, P_2$ of $[a, b]$ s.t.

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{\epsilon}{2}$$

$$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{\epsilon}{2}$$

$$\left. \begin{array}{l} U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{\epsilon}{2} \\ U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{\epsilon}{2} \end{array} \right\} \rightarrow \text{Let } P^* = P_1 \cup P_2, \quad U(P^*, f_i, \alpha) - L(P^*, f_i, \alpha) < \frac{\epsilon}{2}, i=1,2$$

Because $U(P_1, f_1 + f_2, \alpha) \leq U(P_1, f_1, \alpha) + U(P_1, f_2, \alpha)$

$$L(P_1, f_1 + f_2, \alpha) \geq L(P_1, f_1, \alpha) + L(P_1, f_2, \alpha)$$

$$U(P^*, f_1 + f_2, \alpha) - L(P^*, f_1 + f_2, \alpha) :$$

$$\leq U(P^*, f_1, \alpha) + U(P^*, f_2, \alpha) - (L(P^*, f_1, \alpha) + L(P^*, f_2, \alpha))$$

$$< \frac{\epsilon}{2} \times 2$$

$$= \epsilon$$

pf of (d): $|f(x)| \leq M \Rightarrow -M \leq f(x) \leq M$ on $[a, b]$

$$\int_a^b -M d\alpha \leq \int_a^b f d\alpha \leq \int_a^b M d\alpha = M[\alpha(b) - \alpha(a)]$$

$$\Rightarrow \left| \int_a^b f d\alpha \right| \leq M[\alpha(b) - \alpha(a)]$$

6.13

Theorem If $f \in \mathcal{R}(a)$ and $g \in \mathcal{R}(a)$ on $[a, b]$, then (a) $fg \in \mathcal{R}(a)$; (b) $|f| \in \mathcal{R}(a)$ and $|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$.

pf: (a) Take $\phi(t) = t^2$, by theb.11 $f' \in \mathcal{R}(\alpha)$.

Because $4fg = (f+g)^2 - (f-g)^2$, $fg \in \mathcal{R}(\alpha)$.

(b) Take $\phi(t) = |t|$, by theb.11 $|f| \in \mathcal{R}(\alpha)$.

Choose $C = \pm 1$, s.t. $C \int f d\alpha \geq 0$.

Because $Cf \leq |f|$, $|\int f d\alpha| = C \int f d\alpha = \int Cf d\alpha \leq \int |f| d\alpha$

6.7

Theorem Assume α increases monotonically and $\alpha' \in \mathcal{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$.

$$\int_a^b f d\alpha = \int_a^b f(x) \alpha'(x) dx.$$

pf: Because $\alpha' \in \mathcal{R}$, there $\exists$ a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ s.t. $U(P, \alpha') -$

$$L(P, \alpha') < \varepsilon \quad (*)$$

By mean value theorem, for $t_i \in [x_{i-1}, x_i]$, $\Delta \alpha_i = \alpha'(t_i) \cdot \Delta x_i$, $i = 1, \dots, n$.

Then for $s_i \in [x_{i-1}, x_i]$. By 6.7(b), we get $\sum_{i=1}^n |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i < \varepsilon$.

Let $M = \sup |f(x)|$. Because $\sum_{i=1}^n f(s_i) \Delta \alpha_i = \sum_{i=1}^n f(s_i) \alpha'(t_i) \Delta x_i$

$$|\sum_{i=1}^n f(s_i) \Delta \alpha_i - \sum_{i=1}^n f(s_i) \alpha'(s_i) \Delta x_i| = |\sum_{i=1}^n f(s_i) \cdot \alpha'(t_i) \cdot \Delta x_i - \sum_{i=1}^n f(s_i) \alpha'(s_i) \Delta x_i|$$

$$\leq \sum_{i=1}^n |f(s_i)| \cdot |\alpha'(t_i) - \alpha'(s_i)| \cdot \Delta x_i$$

$$\leq M \sum_{i=1}^n |\alpha'(s_i) - \alpha'(t_i)| \cdot \Delta x_i$$

$$< M\varepsilon$$

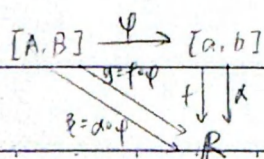
$$\rightarrow |U(P, f, \alpha) - U(P, f\alpha')| \leq M\varepsilon, \quad |L(P, f, \alpha) - L(P, f\alpha')| \leq M\varepsilon.$$

Because $(*)$ remains true if P is replaced by any refinement.

$$|\int_a^b f d\alpha - \int_a^b f(x) \alpha'(x) dx| \leq M\varepsilon, \quad |\int_a^b f d\alpha - \int_a^b f(x) \alpha'(x) dx| \leq M\varepsilon.$$

Because ε is arbitrary taken, $\int_a^b f d\alpha = \int_a^b f(x) \alpha'(x) dx$, $\int_a^b f d\alpha = \int_a^b f(x) \alpha'(x) dx$

Hence, $\int_a^b f d\alpha = \int_a^b f(x) \alpha'(x) dx$.



6.19

Theorem (change of variable) Suppose φ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by $\beta(y) = \alpha(\varphi(y))$, $g(y) = f(\varphi(y))$. Then $g \in \mathcal{R}(\beta)$ and $\int_A^B g d\beta = \int_a^b f d\alpha$.

Note: Take $\alpha(x) = x$, then $\beta = \varphi$. If $\varphi' \in \mathcal{R}$, $\int_A^B f(\varphi(y)) \varphi'(y) dy = \int_a^b f(x) dx$.

6.20

Theorem Let $f \in \mathcal{R}$ on $[a, b]$. For $a \leq x \leq b$, put $F(x) = \int_a^x f(t) dt$. Then F is continuous on $[a, b]$; If f is continuous at x_0 of $[a, b]$, the F is differentiable at x_0 , and $F'(x_0) = f(x_0)$.

pf: Because $f \in \mathcal{R}$, then f is bounded.

Suppose $|f(t)| \leq M$. for $a \leq t \leq b$. If $a \leq x < y \leq b$, then $|F(y) - F(x)| = |\int_x^y f(t) dt| \leq M|y - x| \rightarrow 0$. continuous $\rightarrow$ continuous. (u.c.)

Now suppose f is continuous at x_0 . $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|t - x_0| < \delta$ then $|f(t) - f(x_0)| < \varepsilon$

$$\begin{aligned} |F(x) - F(x_0) - f(x_0)(x - x_0)| &= \left| \int_{x_0}^x f(t) dt - f(x_0)(x - x_0) \right| \\ &= \left| \int_{x_0}^x (f(t) - f(x_0)) dt \right| \\ &\leq \int_{x_0}^x |f(t) - f(x_0)| dt \\ &< \varepsilon |x - x_0| \end{aligned}$$

Hence, $F'(x_0) = f(x_0)$

6.21

(If f and F map $[a, b]$ into $\mathbb{R}^k$) 高维也适用

The fundamental theorem of calculus If $f \in \mathcal{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ s.t. $F' = f$, then $\int_a^b f(x) dx = F(b) - F(a)$.

pf: $\forall \varepsilon > 0$. $\exists P = \{x_0, \dots, x_n\}$ of $[a, b]$ s.t. $U(P, f) - L(P, f) < \varepsilon$.

By mean value theorem, $t_i \in [x_{i-1}, x_i]$, $F(x_i) - F(x_{i-1}) = f(t_i) \Delta x_i$ ($i = 1, \dots, n$).

$$\begin{aligned} \text{Then } \sum_{i=1}^n f(t_i) \Delta x_i &= F(b) - F(a), \quad \varepsilon > \left| \sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f(x) dx \right| = |F(b) - F(a) - \int_a^b f(x) dx| < \varepsilon \end{aligned}$$

Because this holds for every $\varepsilon > 0$. $\int_a^b f(x) dx = F(b) - F(a)$

$$(F(x)G(x))' = f(x)G(x) + F(x)g(x)$$

6.22

Theorem (integration by parts) Suppose F and G are differentiable functions on $[a, b]$. $F' = f \in \mathcal{R}$, $G' = g \in \mathcal{R}$. Then $\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$

pf: $(F(x) \cdot G(x))' = f(x)G(x) + F(x)g(x)$

$$\int_a^b f(x)G(x) + F(x)g(x)dx = F(b)G(b) - F(a)G(a).$$

$$\rightarrow \int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$$

6.25

$f(x) = (f_1(x), f_2(x), \dots, f_k(x))$ 是个向量函数

Theorem If f maps $[a, b]$ into $\mathbb{R}^k$ and if $f \in \mathcal{R}(\alpha)$ for some monotonically increasing function α on $[a, b]$, then $|f| \in \mathcal{R}(\alpha)$, and $|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$. (*)

pf: Let $f_1, \dots, f_k$ be the components of f , then $|f| = (f_1^2 + \dots + f_k^2)^{\frac{1}{2}}$

$$f_1, \dots, f_k \in \mathcal{R}(\alpha) \rightarrow f_1^2, \dots, f_k^2 \in \mathcal{R}(\alpha) \rightarrow f_1^2 + \dots + f_k^2 \in \mathcal{R}(\alpha) \xrightarrow{\text{X is cont.}} (f_1^2 + \dots + f_k^2)^{\frac{1}{2}} \in \mathcal{R}(\alpha)$$

Hence, $|f| \in \mathcal{R}(\alpha)$.

Let $y = (y_1, \dots, y_k)$, where $y_i = \int_a^b f_i d\alpha$. $\rightarrow y = \int_a^b f d\alpha$.

$$|y|^2 = \sum y_i^2 = \sum y_i \int_a^b f_i d\alpha = \int_a^b (\sum y_i f_i) d\alpha.$$

By Schwarz inequality $(\sum_{i=1}^k a_i b_i)^2 \leq (\sum_{i=1}^k a_i^2) (\sum_{i=1}^k b_i^2)$, $\sum y_i f_i \leq |\sum y_i f_i| \leq |y| |f|$

Hence, $|y|^2 \leq |y| \int_a^b |f| d\alpha$.

If $y = 0$, (*) is trivial; If $y \neq 0$, $|y| = |\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$.

6.26

Definition A continuous mapping r of an interval $[a, b]$ into $\mathbb{R}^k$ is called a curve in $\mathbb{R}^k$ or r is a curve on $[a, b]$.

If r is one-to-one, r is called an arc. 圆弧

If $r(a) = r(b)$, r is said to be a closed curve.

Remark: We associate to each partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ and to each curve r on $[a, b]$ the number $\Lambda(P, r) = \sum_{i=1}^n |r(x_i) - r(x_{i-1})|$

Then define the length of r as $\Lambda(r) = \sup \Lambda(P, r)$. which sup is taken over all partitions of $[a, b]$.

If $\Lambda(r) < \infty$, we say that r is rectifiable. 可求长的

6.27

Theorem If r' is continuous on $[a, b]$, then r is rectifiable, and $\Lambda(r) = \int_a^b |r'(t)| dt$.

pf: $a \leq x_{i-1} < x_i \leq b$, then $|r(x_i) - r(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} r'(t) dt \right| \leq \int_{x_{i-1}}^{x_i} |r'(t)| dt$

$\rightarrow \Lambda(P, r) \leq \int_a^b |r'(t)| dt$ for every partition P of $[a, b]$.

$\rightarrow \Lambda(r) \leq \int_a^b |r'(t)| dt$

Because r' is u.c. on $[a, b]$. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|s - t| < \delta$ then $|r'(s) - r'(t)| < \varepsilon$.

Let $P = \{x_0, \dots, x_n\}$ with $\Delta x_i < \delta$ for all i . If $x_{i-1} \leq t \leq x_i$, $|r'(t)| \leq |r'(x_i)| + \varepsilon$.

$\int_{x_{i-1}}^{x_i} |r'(t)| dt \leq |r'(x_i)| \Delta x_i + \varepsilon \Delta x_i$

$$= \left| \int_{x_{i-1}}^{x_i} [r'(t) + r'(x_i) - r'(t)] dt \right| + \varepsilon \Delta x_i$$

$$\leq \left| \int_{x_{i-1}}^{x_i} r'(t) dt \right| + \left| \int_{x_{i-1}}^{x_i} [r'(x_i) - r'(t)] dt \right| + \varepsilon \Delta x_i$$

$$\leq |r(x_i) - r(x_{i-1})| + \varepsilon \Delta x_i + \varepsilon \Delta x_i$$

$$= |r(x_i) - r(x_{i-1})| + 2\varepsilon \Delta x_i$$

$$\rightarrow \int_a^b |r'(t)| dt \leq \Lambda(P, r) + 2\varepsilon(b-a) \leq \Lambda(r) + 2\varepsilon(b-a)$$

Because ε is arbitrary, $\int_a^b |r'(t)| dt \leq \Lambda(r)$.

Hence, $\Lambda(r) = \int_a^b |r'(t)| dt$.

Sequences and Series of Function 函数序列与级数

7.1 Definition (Pointwise convergence) Suppose $\{f_n\}$, $n=1, 2, 3, \dots$, is a sequence of functions defined on a set E . ^{每个 x 都有一个对应的 ε} $\forall x \in E, \forall \varepsilon > 0, \exists N$ s.t. if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$.

We say $\{f_n\}$ converges to f pointwise, and f is the limit function / or limit.

Similarly, if $\sum f_n(x)$ converges pointwise for every $x \in E$, then we define

$f(x) = \sum_{n=1}^{\infty} f_n(x)$ ($x \in E$), the function f is called the sum of the series $\sum f_n$.

注: 如果我们需证明 f 在 x 连续 i.e. $\lim_{t \rightarrow x} f(t) = f(x) \Leftrightarrow \lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t)$.

例: ① 连续函数构成的序列的级数可能有不连续的和.

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \quad (x \in \mathbb{R}, n=0, 1, 2, \dots)$$

$$\text{Let } f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

$$\text{For } x=0, f_n(0) = 0 \rightarrow f(0) = 0$$

$$\text{For } x \neq 0, f(x) = \frac{x^2}{1-x^2} = \frac{x^2(1+x^2)}{(1+x^2)-1} = 1+x^2 \rightarrow f(x) = \begin{cases} 0, & x=0 \\ 1+x^2, & x \neq 0 \end{cases}$$

Hence, a convergent series of continuous functions may have a discontinuous sum.

② 连续函数在逐点收敛下的极限函数可能不连续.

$$f_n: [0, 1] \rightarrow \mathbb{R}, f_n(x) = x^n$$

$$\text{i) } x=0, f_n(x) = 0, \text{ then } f(x) = 0.$$

$$\text{ii) } 0 < x < 1.$$

$$\frac{x^{n+1}}{x^n} = x < 1 \rightarrow x^n \downarrow, x^n > 0, \lim_{n \rightarrow \infty} x^n \text{ exists.}$$

$$\text{Let } L = \lim_{n \rightarrow \infty} x^n = \lim_{n \rightarrow \infty} x^{n+1} = x \lim_{n \rightarrow \infty} x^n = x \cdot L, x \in (0, 1) \rightarrow L = 0.$$

$$\text{Hence, } f(x) = 0$$

$$\text{or } \forall \varepsilon > 0, \exists N \text{ s.t. if } n \geq N, \text{ then } x^n < \varepsilon. \Leftrightarrow \ln x^n < \ln \varepsilon \Leftrightarrow n \ln x < \ln \varepsilon$$

$$\Leftrightarrow n > \frac{\ln \varepsilon}{\ln x}$$

$$\text{Let } N > \frac{\ln \varepsilon}{\ln x}, \text{ then } n \geq N > \frac{\ln \varepsilon}{\ln x}.$$

$$\text{iii) } x=1, f_n(x) = 1, \text{ then } f(x) = 1$$

Hence, $f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$. The concept of pointwise convergence is therefore too weak to allow for continuity properties to carry over to limit functions.

③ 如果 $\int_a^b f_n = L$ 对于所有 n , 且 f_n 逐点收敛到 f , 不代表 $\int_a^b f = L$.

$$f_n: [0, 1] \rightarrow \mathbb{R}, f_n(x) = \begin{cases} 2n & x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^1 f_n = 2n \times \frac{1}{2n} = 1. \text{ But } \int_0^1 f = \lim_{n \rightarrow \infty} \int_0^1 f_n = 0.$$

$$\text{Hence, } \lim_{n \rightarrow \infty} \int_a^b f_n \neq \int_a^b \lim_{n \rightarrow \infty} f_n$$

④ f'_n 不一定收敛到 f' .

$$f_n(x) = \frac{\sin nx}{n} \quad (x \in \mathbb{R}, n=1, 2, 3, \dots) \rightarrow f'_n(x) = \cos nx.$$

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0 \rightarrow f'(x) = 0.$$

Because $f'_n(0) = 1 \rightarrow +\infty$ as $n \rightarrow \infty$. $\{f'_n\}$ does not converge to f' .

$$\textcircled{5} \lim_{n \rightarrow \infty} \int_a^b f_n \neq \int_a^b \lim_{n \rightarrow \infty} f_n.$$

$$f_n(x) = n^2 x (1-x^2)^n \quad (0 \leq x \leq 1, n=1, 2, 3, \dots).$$

$$\int_0^1 x (1-x^2)^n dx = \frac{1}{2n+2} \rightarrow \int_0^1 f_n(x) dx = \frac{n^2}{2n+2} \rightarrow \lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = +\infty.$$

$$\lim_{n \rightarrow \infty} f_n(x) = 0 \rightarrow \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx = 0.$$

$$-\frac{1}{2} \int_0^1 (1-x^2)^n d(1-x^2) = -\frac{1}{2} \int_1^0 t^n dt = \frac{1}{2} \int_0^1 t^n dt = \frac{1}{2} \left[\frac{t^{n+1}}{n+1} \right]_0^1 = \frac{1}{2(n+2)}$$

7.7

Definition We say that a sequence of functions $\{f_n\}$, $n=1, 2, 3, \dots$, converges uniformly on E to a function f if for every $\varepsilon > 0$ there is an integer N s.t. $n \geq N$ implies $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in E$.

We say that the series $\sum f_n(x)$ converges uniformly on E if the sequence $\{S_n\}$ of partial sums defined by $\sum_{i=1}^n f_i(x) = S_n(x)$ converges uniformly on E .

Remark: If a sequence $f_n: X \rightarrow Y$ converges pointwise (or uniformly) to a function $f: X \rightarrow Y$, then the restrictions $f_n|_E: E \rightarrow Y$ of f to some subset E of X will also converge pointwise (or uniformly) to $f|_E$.

eg. $f_n: [0, 1] \rightarrow \mathbb{R}, f_n(x) = \frac{x}{n}$ converges uniformly to $f(x) = 0$.

We have to show that for $\forall \varepsilon > 0, \exists N$ s.t. $|f_n(x) - f(x)| < \varepsilon$ for every $x \in [0, 1]$ and every $n \geq N$.

First, let's fix an $\varepsilon > 0$. Then for any $x \in [0, 1]$ and $n \geq N$, we have

$$|f_n(x) - f(x)| = \left| \frac{x}{n} - 0 \right| = \frac{x}{n} \leq \frac{1}{n} \leq \frac{1}{N}.$$

Campus 不一致收敛: $\exists \varepsilon > 0$, s.t. $\forall N \in \mathbb{N}, \exists n > N, x \in E$ s.t. $|f_n(x) - f(x)| \geq \varepsilon$.

逐点收敛: 先固定一个 x 和一个 ε . 再去求 N . N 即受 x 的约束, 又受 ε 的约束.

一致收敛: N 的取值仅与 ε 有关, 与 x 无关. 在 N 值之后的所有 f_n , 对于所有的 x 值, 其函数值与极限函数的距离都在 ε 内. every $n > N$ and $x \in X$.

$(f_n)_{n=1}^{\infty}$ converges uniformly to f on X if for every $\varepsilon > 0 \exists N > 0$ s.t. $d_Y(f_n(x), f(x)) < \varepsilon$ for

6160
- 619
619

No.

Date

(Note that this choice of N doesn't depend on

$|f_n(x) - f(x)| < \varepsilon$ for all $n > N$ and $x \in [0, 1]$.

Not converge uniformly to f : $\exists \varepsilon > 0$, s.t. for any $N \in \mathbb{N}$, $\exists n > N$ and $x \in E$ with

$$|f_n(x) - f(x)| \geq \varepsilon.$$

x 也是存在即可

eg. $f_n: (0, 1) \rightarrow \mathbb{R}$, $f_n(x) = x^n$, $n = 1, 2, \dots$, not c.u. to $f = 0$.

Take $\varepsilon = \frac{1}{2}$. For any $N \in \mathbb{N}$, let $n = N+1$ and $x = (1 - \frac{1}{N+1})^{\frac{1}{N+1}}$

$$|f_n(x) - 0| = (1 - \frac{1}{N+1})^{\frac{1}{N+1} \cdot (N+1)} = 1 - \frac{1}{N+1} \geq 1 - \frac{1}{2} = \frac{1}{2} = \varepsilon.$$

Hence, f_n does not converge uniformly to 0.

eg. $f_n: [0, 1] \rightarrow \mathbb{R}$, $f_n(x) = n^2 x(1-x)^n$, $n = 1, 2, \dots$, not c.u. to $f = 0$.

Take $\varepsilon = \frac{1}{e}$. For any $N \in \mathbb{N}$, let $n > N$ and $x = \frac{1}{n}$.

$$|f_n(x) - 0| = n^2 \frac{1}{n} (1 - \frac{1}{n})^n \rightarrow n \cdot \frac{1}{e} > \frac{1}{e}$$

$|f_n(x)|$ 随着 n 增大趋向无穷大, 而不是 0.

7.8 c.u. of Cauchy

Theorem The sequence of functions $\{f_n\}$, defined on E , converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N s.t. $m > N$, $n > N$, $x \in E$ implies $|f_n(x) - f_m(x)| \leq \varepsilon$. (*)

p.f. $\Rightarrow$ Suppose $\{f_n\}$ converges uniformly on E , and let f be the limit function.

Then $\exists N$ s.t. $n > N$, $x \in E$, then $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2}$.

Hence, for $n > N$, $m > N$, $x \in E$, $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \varepsilon$.

$\Leftarrow$ Cauchy condition holds then the sequence $\{f_n(x)\}$ converges for every x .

Let $\varepsilon > 0$ be given, and choose N s.t. (*) holds.

Fix n , let $m \rightarrow \infty$, this gives $|f_n(x) - f(x)| \leq \varepsilon$ for every $n > N$ and every $x \in E$. i.e. $\{f_n\}$ converges uniformly on E .

7.9 C.U.

Theorem Suppose $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, ($x \in E$). Put $M_n = \sup_{x \in E} |f_n(x) - f(x)|$. Then $f_n \rightarrow f$ uniformly on E if and only if $M_n \rightarrow 0$ as $n \rightarrow \infty$.

pf: $\sup_{x \in E} |f_n(x) - f(x)| \rightarrow 0$ as $n \rightarrow \infty \Leftrightarrow f_n(x) \rightarrow f$ at each x . $\Leftrightarrow \exists N$ s.t. $m \geq N, n \geq N, x \in E$ s.t. $|f_n(x) - f_m(x)| \leq \varepsilon$. Then this is equal to The 7.8

7.10 C.U. of $\sum f_n$

Theorem Suppose $\{f_n\}$ is a sequence of functions defined on E , and suppose $|f_n(x)| \leq M_n$ ($x \in E, n=1, 2, 3, \dots$). Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

pf: $\sum f_n$ C.U. $\Leftrightarrow \{S_n\}$ C.U. $S_n = \sum_{k=1}^n f_k \Leftrightarrow \forall \varepsilon > 0, \exists N$ s.t. $|S_n(x) - S_m(x)| \leq \varepsilon, \forall n, m \geq N, x \in E$.

$\Leftrightarrow \forall \varepsilon > 0, \exists N$ s.t. $|\sum_{k=1}^n f_k - \sum_{k=1}^m f_k| = |\sum_{k=m+1}^n f_k| \leq \varepsilon, \forall n, m \geq N, x \in E$.

$|\sum_{k=m+1}^n f_k(x)| \leq \sum_{k=m+1}^n |f_k(x)| \leq \sum_{k=m+1}^n M_k \leq \varepsilon, (x \in E)$, provided m, n are large enough.

u.c. follows from The. 7.8

7.12

可以推广到任意度量空间

Theorem Let $K \subset \mathbb{R}$ or $\mathbb{C}$ and $f_n: K \rightarrow \mathbb{R}$ (or $\mathbb{C}$) continuous functions which converge uniformly to $f: K \rightarrow \mathbb{R}$ (or $\mathbb{C}$). Then the function f is continuous.

pf: Let $x \in K, \forall \varepsilon > 0$. (在 x 连续或者在定义域上连续都可以通过一致连续传递)

Because f_n u.c. to $f, \exists N$ s.t. all $x \in K, |f_N(x) - f(x)| < \frac{\varepsilon}{3}$.

Because f_N are continuous, $\exists \delta > 0$ s.t. $|x - y| < \delta$, then $|f_N(y) - f_N(x)| < \frac{\varepsilon}{3}$ for all $y \in K$. (Corresponding to x and ε).

Then $|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|$
 $< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

Hence, f is continuous.

7.13 C.U.

Theorem Suppose K is compact, and

(a) $\{f_n\}$ is a sequence of continuous functions on K ,

(b) $\{f_n\}$ converges pointwise to a continuous function f on K ,

向下传递

(c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K, n=1, 2, 3, \dots$

Corollary Then $f_n \rightarrow f$ uniformly on K .

pf: Let $g_n = f_n - f$, then g_n is continuous, $g_n \rightarrow 0$ pointwise, $g_n \geq g_{n+1}$.

Then we want to prove $g_n \rightarrow 0$ uniformly on K .

$\forall \varepsilon > 0$, let $K_n = \{x \in K : g_n(x) \geq \varepsilon\} \rightarrow K_n = g_n^{-1}([\varepsilon, \infty))$

Because $[\varepsilon, \infty)$ is closed. (All limit points are in it), g_n is continuous.

$\rightarrow K_n$ is closed.

Because K is compact, then K_n is compact.

Because $g_n \geq g_{n+1}$, then $K_n \supset K_{n+1}$.

$\forall x \in K$. Since $g_n(x) \rightarrow 0$, $x \notin K_n$ if n is sufficiently large.

$\rightarrow \exists N$ s.t. $K = \bigcap_{n=1}^{\infty} K_n = \emptyset$ for $n \geq N \rightarrow \exists N$ s.t. $K_n = \emptyset$ for $n \geq N$

$\rightarrow \exists N$ for all $x \in X$, $g_n(x) < \varepsilon$ if $n \geq N$

eg. If the compactness is not in it. eg. $f_n(x) = \frac{1}{n(x+1)}$ ($0 < x < 1$; $n=1, 2, 3, \dots$).

$f_n(x) \rightarrow 0$ monotonically in $(0, 1)$, but the convergence is not uniform.

(Let $\varepsilon = \frac{1}{2}$, $N \geq 1$, let $n = N$, $x = \frac{1}{N}$, $|f_n(x) - 0| = \frac{1}{N \cdot \frac{1}{N} + 1} = \frac{1}{2}$).

$g_n = f_n - f$

$g_n = f_n - f$. g_n 连续, $g_n \rightarrow 0$ 逐点, $g_n \geq g_{n+1}$

To prove: $\forall \varepsilon > 0$, $\exists N$ s.t. $n \geq N$, $|g_n| < \varepsilon$.

$\forall \varepsilon > 0$. $K_n = \{x \in K : g_n(x) \geq \varepsilon\}$ K_n 在某 n 开始为空.

$K_n = g_n^{-1}([\varepsilon, \infty))$ K 紧 $\rightarrow K_n$ 紧

闭 逐 闭.

$\because g_n \rightarrow 0$. $\exists N$ s.t. $K_n = \emptyset$ for all $n \geq N$.

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$$\frac{1}{N(x+1)}$$

$$\|x+y\| \leq \|x\| + \|y\| \leq \|x\| + \|y\|$$

Definition Let X be a vector space over the field K . A norm on X is a function $\|\cdot\|: X \rightarrow [0, \infty)$, with the properties

i) $\|\lambda x\| = |\lambda| \|x\|$; ii) $\|x+y\| \leq \|x\| + \|y\|$; iii) $\|x\| = 0 \Leftrightarrow x = 0$.
for all $\lambda \in K$ and $x, y \in X$. The pair $(X, \|\cdot\|)$ is said to be a normed space.

Remark: ① Every normed space is also a metric space with respect to the induced metric $d(x, y) := \|x - y\|$.
(i) $d(\lambda x, \lambda y) = \|\lambda x - \lambda y\| = |\lambda| \|x - y\| = |\lambda| d(x, y)$
(ii) $d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y)$

② Convergence in a normed space is defined by the metric $d: X_n \rightarrow X$ in $(X, \|\cdot\|)$ if $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.
(iii) $d(x, y) = 0 \Leftrightarrow x = y, x - y = 0$

③ If (X, d) is complete, we say $(X, \|\cdot\|)$ a Banach space.

eg. The absolute value $|\cdot|$ in $\mathbb{R}$ can be generalized to a norm in $\mathbb{R}^d$ by putting for $x = (x^1, \dots, x^d) \in \mathbb{R}^d$, $\|x\|_p = \left(\sum_{i=1}^d |x^i|^p\right)^{\frac{1}{p}}$ with $1 \leq p < \infty$, or by $\|x\|_\infty = \max_{i=1, \dots, d} |x^i|$

Lemma Let $(X, \|\cdot\|)$ be a normed space and let $(x_n)_{n \in \mathbb{N}} \subset X$ converge to $x \in X$. Then $\lim_{n \rightarrow \infty} \|x_n\| = \|x\|$ i.e. the norm is continuous.

pf: $|\|x_n\| - \|x\|| \leq \|x_n - x\|$ and $\|x_n - x\| \rightarrow 0$ (because $(x_n)_{n \in \mathbb{N}}$ converges to x).

$$\|x_n\| - \|x\| \rightarrow 0 \text{ i.e. } \|x_n\| \rightarrow \|x\|, \lim_{n \rightarrow \infty} \|x_n\| = \|x\|$$

Definition $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isomorphic, write $X \simeq Y$, if $\exists$ a bijective linear map $T: X \rightarrow Y$ and $C_1 > 0, C_2 > 0$ s.t. $C_1 \|x\|_X \leq \|T(x)\|_Y \leq C_2 \|x\|_X$ for $\forall x \in X$; and are isometrically isomorphic if in addition T preserves the norm, $\|T(x)\|_Y = \|x\|_X$ for $\forall x \in X$. T is then called an isometric isomorphism from X onto Y and write $X \equiv Y$.

Definition Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are equivalent, write $\|\cdot\|_1 \sim \|\cdot\|_2$ if there exist constants $0 < C_1 \leq C_2$ s.t. $C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1$ for all $x \in X$.

Remark: ① $C_1 \|x_n - x\|_1 \leq \|x_n - x\|_2 \leq C_2 \|x_n - x\|_1$ i.e. $\|x_n - x\|_1 \rightarrow 0 \Leftrightarrow \|x_n - x\|_2 \rightarrow 0$

一致度量: $d_r(f, g) = \sup_{x \in X} |f(x) - g(x)|$

② If $\|\cdot\|_1 \sim \|\cdot\|_2$, $\|\cdot\|_2 \sim \|\cdot\|_3$, then $\|\cdot\|_1 \sim \|\cdot\|_3$.

证法. Theorem If $\dim(X) = n < \infty$, then any two norms on X are equivalent. 等价范数

pf: Let $\{u_i\}_{i=1}^n$ be a base of X and define $\forall u = \sum_{i=1}^n x_i u_i \in X$, $\|u\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$
 $\|u\|_1 = \left\| \sum_{i=1}^n x_i u_i \right\| \leq \sum_{i=1}^n \|x_i u_i\| = \sum_{i=1}^n |x_i| \|u_i\| \leq \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} \left(\sum_{i=1}^n \|u_i\|^2\right)^{1/2} = C_2 \|u\|_2$

It suffices to show $\rightarrow \|u\| \leq C_2 \|u\|_2, \forall u \in X$.

that any norm $\|\cdot\|$ Define $f: (X, \|\cdot\|_2) \rightarrow \mathbb{R}, f(u) = \|u\| = \sum_{i=1}^n x_i u_i$

on X is Claim: f is continuous on $(X, \|\cdot\|_2)$ ($\forall x, y \in X, \therefore |f(x) - f(y)| = \|\|x\| - \|y\|\| \leq \|x - y\|_2 \leq C_2 \|x - y\|_2$) $\therefore f$ is continuous

$$\tilde{S} = \left\{ \sum_{i=1}^n x_i u_i : \sum_{i=1}^n x_i^2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} = \left\| \sum_{i=1}^n x_i u_i \right\|_2 = 1 \right\}$$

有界闭连续 $\rightarrow$ 上确界可取

$$S = \left\{ \sum_{i=1}^n x_i e_i = (x_1, \dots, x_n) : \sum_{i=1}^n x_i^2 = 1 \right\} \quad T(S) = \tilde{S}$$

Let $T: \mathbb{R}^n \rightarrow X, T\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i u_i$, obviously, T is linear bijection.

For $y \in X, \|u\| \leq \left\| \frac{y}{\|y\|_2} \right\| \rightarrow \|y\| \geq \|u\| \|y\|_2$ By claim, T is continuous.

$$(\forall y \in X, y \neq 0, \frac{y}{\|y\|_2} \in \tilde{S}, \left\| \frac{y}{\|y\|_2} \right\|_2 = \frac{\|y\|_2}{\|y\|_2} = 1)$$

Because T is continuous, S is compact in $\mathbb{R}^n$, then $\tilde{S}$ is compact in $(X, \|\cdot\|_2)$

f attains its minimum at some $u_0 \in \tilde{S}$ i.e. $\|u_0\| \leq \|u\|, \forall u \in \tilde{S}$.

$$\therefore \forall u \in X, \|u\|_2 \leq \frac{1}{\|u_0\|} \|u\| = C_1 \|u\|$$

Hence, $\|\cdot\| \sim \|\cdot\|_2$.

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所有连续且有界的函数的集合

Definition If X is a metric space, $\mathcal{B}(X)$ will denote the set of all complex-valued, continuous, bounded functions with domain X . $(\mathcal{B}(X), \|\cdot\|)$ is a normed vector space.

Remark: $f_n \rightarrow f$ w.r.t. the metric of $\mathcal{B}(X)$ iff $f_n \rightarrow f$ uniformly on X .

在连续函数空间 $\mathcal{B}(X)$ 上的度量收敛等价于一致收敛 在 $\mathcal{B}(X)$ 中, 我们关心的是收敛序列收敛的全局估计

Definition Let K be a set and $f: K \rightarrow \mathbb{R}$ (or $\mathbb{C}$) a function.

$$\|f\|_K = \sup \{|f(x)| : x \in K\}$$

Lemma $\|\cdot\|_K$ is a norm on the vector space of bounded real (resp. complex) valued functions

on K . $\mathcal{B}(X)$ is the space of all $f(x)$ with $\|f\| = \sup_{x \in X} |f(x)| < \infty$ KOKUYO

$\rightarrow \mathcal{B}(X)$ is complete.

Theorem $f_n: K \rightarrow \mathbb{R}$ (or $\mathbb{C}$) converges uniformly to $f: K \rightarrow \mathbb{R}$ (resp. $\mathbb{C}$) iff

$$\lim_{n \rightarrow \infty} \|f_n - f\|_K = 0.$$

pf: f_n converges uniformly to $f \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall x \in K: |f_n(x) - f(x)| < \varepsilon.$

$$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N: \sup\{|f_n(x) - f(x)| : x \in K\} < \varepsilon.$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N: \|f_n - f\|_K < \varepsilon.$$

$$\Leftrightarrow \lim_{n \rightarrow \infty} \|f_n - f\|_K = 0.$$

7.16

Theorem Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \dots$, and suppose $f_n \rightarrow f$ uniformly on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$ on $[a, b]$, and $\int_a^b f d\alpha = \lim_{n \rightarrow \infty} \int_a^b f_n d\alpha.$

pf: For $x \in [a, b]$, let $\varepsilon_n = \sup |f_n(x) - f(x)| \rightarrow f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n.$

$$\rightarrow \int_a^b (f_n - \varepsilon_n) d\alpha \leq \int_a^b f d\alpha \leq \int_a^b (f_n + \varepsilon_n) d\alpha. (*)$$

$$\rightarrow 0 \leq \int_a^b f d\alpha - \int_a^b f_n d\alpha \leq 2\varepsilon_n [\alpha(b) - \alpha(a)] \quad \text{两端积分相减}$$

Then because $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, $\int_a^b f d\alpha = \int_a^b f_n d\alpha$. Thus $f \in \mathcal{R}(\alpha).$

Then from (*) we get $|\int_a^b f d\alpha - \int_a^b f_n d\alpha| \leq \varepsilon_n [\alpha(b) - \alpha(a)].$

$$\text{Hence, } \int_a^b f d\alpha = \lim_{n \rightarrow \infty} \int_a^b f_n d\alpha. \quad \int_a^b f d\alpha - \int_a^b f_n d\alpha = \int_a^b (f - f_n) d\alpha \leq \int_a^b (f_n + \varepsilon_n - f) d\alpha \leq \varepsilon_n [\alpha(b) - \alpha(a)]$$

注: $f_n(x)$ 可积, 但 f 不可积的例子.

$$Q = \{r_1, r_2, \dots\}. \quad f_n(x) = \begin{cases} 1 & x = r_1, \dots, r_n \\ 0 & x \notin \{r_1, \dots, r_n\} \end{cases} \rightarrow f = \begin{cases} 1 & x \in Q \\ 0 & x \notin Q \end{cases}$$

有限个不连续点, 故可积

Corollary If $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ and if $f(x) = \sum_{n=1}^{\infty} f_n(x)$ ($a \leq x \leq b$), the series converging uniformly on $[a, b]$, then $\int_a^b f d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n d\alpha.$ 对于函数级数的推广.

Proposition (Newton's approximation). Let X be a subset of $\mathbb{R}$, let $x_0 \in X$ be a limit point of X , let $f: X \rightarrow \mathbb{R}$ be a function, and let L be a real number. Then the following statements are logically equivalent:

(a) f is differentiable at x_0 on X with derivative L .

(b) For every $\varepsilon > 0$, there $\exists \delta > 0$ s.t. $|f(x) - (f(x_0) + L(x - x_0))| \leq \varepsilon |x - x_0|$ whenever $x \in X$ and $|x - x_0| \leq \delta$.

Theorem Let $I = [a, b]$ be a bounded interval in $\mathbb{R}$. Let $f_n: I \rightarrow \mathbb{R}$ be differentiable functions. Assume that

(i) there exists a $z \in I$ for which $f_n(z)$ converges.

(ii) the sequence of derivatives (f'_n) converges uniformly on I .

Then the sequence (f_n) converges uniformly on I to a differentiable function f and we have $f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$ for all $x \in I$.

pf: Let $g(x)$ be the limit of $f'_n(x)$. $\forall \eta > 0$.

Because (f'_n) converges uniformly,

$\exists N \in \mathbb{N}$, s.t. $\forall n, m \geq N$ then $\sup \{|f'_n(x) - f'_m(x)|\} < \eta$ for all $x \in I$. ①

Let $m \rightarrow +\infty$, $\sup \{|f'_n(x) - g(x)|\} < \eta$ for all $x \in I$.

For all $x \in I$, $n, m \in \mathbb{N}$, by mean value theorem,

$$|f_n(x) - f_m(x) - (f_n(z) - f_m(z))| \leq |x - z| \sup_{\xi \in I} |f'_n(\xi) - f'_m(\xi)| \quad (*)$$

$$\Rightarrow |f_n(x) - f_m(x)| - |f_n(z) - f_m(z)| \leq |x - z| \sup_{\xi \in I} |f'_n(\xi) - f'_m(\xi)|$$

$$\Rightarrow |f_n(x) - f_m(x)| \leq |f_n(z) - f_m(z)| + |x - z| \sup_{\xi \in I} |f'_n(\xi) - f'_m(\xi)|$$

Hence, because $f_n(z)$ converges, (f'_n) converges uniformly, then (f_n) converges uniformly.

Because $(f_n)_{n \in \mathbb{N}}$ are continuous, and (f_n) converges uniformly, then f is continuous

For $(*)$, let $n = N$, $m \rightarrow +\infty$, and by ①:

$$|f_N(x) - f(x) - f_N(z) + f(z)| < |x - z| \eta \quad (2)$$

For N , which depends only on η , $\exists \delta > 0$ then $|x - z| < \delta$

$$|f_N(x) - f_N(z) - (x - z)f'_N(x)| \leq \eta |x - z| \quad (f \text{ is continuous})$$

$$\text{Then } |f(x) - f(z) - g(x)(x - z)|$$

$$= |[f'_n(x)(x-z) - g(x)(x-z)] - [f'_n(x) - f(x) - f'_n(z) + f(z)] + [(f'_n(x) - f'_n(z)) - (x-z)f'_n(x)]|$$

By 0.0.0

$$< 3\eta|x-z|.$$

Because x can be taken arbitrarily in I and z satisfies $|x-z| < \delta$.

then follows from our characterization of differentiability, $f'(x)$ exists and $f'(x) = g(x) = \lim_{n \rightarrow \infty} f'_n(x)$ for all $x \in I$.

Theorem Let K be $\mathbb{R}$ or $\mathbb{C}$, X be a set. Let $\mathcal{F}_b(X; K)$ be the collection of all functions $f: X \rightarrow K$ that are bounded. i.e. $\sup_{x \in X} |f(x)| < \infty$. Then $\mathcal{F}_b(X; K)$ is complete with the supremum norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$.
 度量不同, 会影响 Cauchy 序列 $\{f_n, f_m\}$ 的计算

Remark: Convergence of functions in the supremum norm i.e. $\|f_n - f\|_\infty \rightarrow 0$, is known as uniform convergence, since it implies that given any $\varepsilon > 0$, $\exists N$ s.t. $\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ i.e. $|f_n(x) - f(x)| < \varepsilon$ for every $x \in X$.

pf: Suppose $(f_k)_{k=1}^\infty$ is Cauchy in $\mathcal{F}_b(X; K)$. *x 可以任取*

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \|f_n - f_m\|_\infty = \sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon \text{ for all } n, m \geq N. (*)$$

For pointwisely, fixed $x \in X$, the sequence $(f_k(x))$ is Cauchy in K , and so converges (because K is complete)

$$\text{Hence, } f(x) = \lim_{k \rightarrow \infty} f_k(x).$$

By (*), we get $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m \geq N$.

Then let $m \rightarrow \infty$, we get $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$.

$$\rightarrow \sup_{x \in X} |f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N. (**)$$

Hence, f_n converges uniformly to f on X .

Last is to show that limiting function f is bounded.

By (**), because f_N is bounded, then $\|f_N - f\|_\infty < \varepsilon$.

在 $C([a, b])$ 中 $\rightarrow$ 一致收敛 在 $C^1([a, b])$ 中 $\rightarrow$ 一致收敛 (一致可微)

Theorem The space $C^1([a, b])$ of all continuously differentiable functions on $[a, b]$ is complete with the C^1 norm $\|f\|_{C^1} = \|f\|_{\infty} + \|f'\|_{\infty}$.

pf: Suppose $(f_n)_{n=1}^{\infty} \in C^1([a, b])$ be Cauchy.

$\forall \varepsilon > 0, \exists N$ s.t. $\|f_n - f_m\|_{\infty} + \|f'_n - f'_m\|_{\infty} < \varepsilon$ for all $n, m \geq N$ (**)

Hence, $(f_n)_{n=1}^{\infty}$ and $(f'_n)_{n=1}^{\infty}$ are both Cauchy sequences in $C([a, b])$ with $\|\cdot\|_{\infty}$.

Because $C([a, b])$ is complete, then $\exists f, g \in C([a, b])$ s.t. $f_n \rightarrow f$ and $f'_n \rightarrow g$.

Then we need to show $g = f'$.

$$f_n(x) = f_n(a) + \int_a^x f'_n(t) dt, x \in [a, b]$$

$$\text{Because } \left| \int_a^x f'_n(t) dt - \int_a^x g(t) dt \right| \leq \int_a^x |f'_n(t) - g(t)| dt \leq (b-a) \|f'_n - g\|_{\infty}$$

As $n \rightarrow \infty, \|f'_n - g\|_{\infty} \rightarrow 0$, then $\int_a^x f'_n(t) dt \rightarrow \int_a^x g(t) dt$ uniformly on $[a, b]$

$$\text{For } \textcircled{A}, \text{ after taking limits, } \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} f_n(a) + \lim_{n \rightarrow \infty} \int_a^x f'_n(t) dt$$

$$\rightarrow f(x) = f(a) + \int_a^x g(t) dt$$

$$\text{Hence, } g = f' \rightarrow f'_n \rightarrow f'$$

For (**), take $m \rightarrow \infty, \|f_n - f\|_{\infty} + \|f'_n - f'\|_{\infty} < \varepsilon, n \geq N$

Because $f'_n \rightarrow f'$, then $f_n \rightarrow f$ in $C^1([a, b])$.

导数一致有界

Proposition Assume that f_n are differentiable on $[a, b]$ and that $|f'_n(x)| \leq M$ for some $M > 0$ and all n and $x \in [a, b]$. If $f_n \rightarrow f$ pointwise, then $f_n \rightarrow f$ uniformly.

pf: $\forall \varepsilon > 0$. Take a partition $P = \{a = x_0 < x_1 < \dots < x_l = b\}$ s.t. $x_i - x_{i-1} < \frac{\varepsilon}{3M}, i = 1, \dots, l$.

Because $f_n \rightarrow f$ pointwise, for each $k \in \{0, \dots, l\}$, $\{f_n(x_k)\}_{n=1}^{\infty}$ converges.

$\exists N \in \mathbb{N}$ s.t. $|f_n(x_k) - f_m(x_k)| < \frac{\varepsilon}{3}, k = 0, 1, \dots, l$ for all $n, m \geq N$.

Let $x \in [a, b]$, suppose $x \in [x_{k-1}, x_k]$, for any n ,

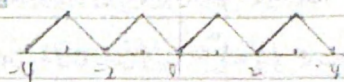
$$|f_n(x) - f_n(x_k)| \leq M \cdot \frac{\varepsilon}{3M} = \frac{\varepsilon}{3}$$

$$\text{Then, for } m, n \geq N, |f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(x)| < \varepsilon$$

Hence, $f_n \rightarrow f$ uniformly.

Theorem There exists a real continuous function on the real line which is nowhere differentiable. i.e. $f'(x)$ does not exist.

pf: Define $\varphi(x) = |x|$ ($-1 \leq x \leq 1$).



Then extend the definition of $\varphi(x)$ to all real x by $\varphi(x+2) = \varphi(x)$.

By the graph of $\varphi(x)$, for all s and t , $|\varphi(s) - \varphi(t)| \leq |s - t|$.

Define $f(x) = \sum_{n=0}^{\infty} (\frac{3}{4})^n \varphi(4^n x)$. 是 φ 的自变量

Because $\varphi \in [0, 1]$, $(\frac{3}{4})^n \varphi(4^n x) \leq (\frac{3}{4})^n$.

Because $\sum (\frac{3}{4})^n$ converges, the ^{series} converges uniformly on $\mathbb{R}$.

Because these series are continuous, f is continuous on $\mathbb{R}$.

Then fix x and a positive integer m . Put $\delta_m = \pm \frac{1}{2} \cdot 4^{-m}$.

We ^{can choose that} there is no integer lies between $4^m x$ and $4^m(x + \delta_m)$. (Because $|4^m x - 4^m(x + \delta_m)| = |-4^m \delta_m| = \frac{1}{2}$.)

Define $r_n = \frac{\varphi(4^n(x + \delta_m)) - \varphi(4^n x)}{\delta_m}$.

Because $4^m \delta_m$ is an even integer, $T=2$, then $r_n = 0$ for $n > m$.

For $0 \leq n \leq m$, $|r_n| = \frac{|\varphi(4^n(x + \delta_m)) - \varphi(4^n x)|}{\delta_m} \leq \frac{|4^n \delta_m|}{|\delta_m|} = |4^n| = 4^n$.

Because $|r_m| = \frac{|\varphi(4^m(x + \frac{1}{2})) - \varphi(4^m x)|}{\delta_m} = \frac{|\frac{1}{2}|}{\pm \frac{1}{2} \cdot 4^{-m}} = 4^m$,

$|\frac{f(x + \delta_m) - f(x)}{\delta_m}| = |\frac{1}{\delta_m} (\sum_{n=0}^{\infty} (\frac{3}{4})^n \varphi(4^n(x + \delta_m)) - \sum_{n=0}^{\infty} (\frac{3}{4})^n \varphi(4^n x))|$

$$= |\frac{1}{\delta_m} \sum_{n=0}^m (\frac{3}{4})^n [\varphi(4^n(x + \delta_m)) - \varphi(4^n x)]|$$

$$= |\sum_{n=0}^m (\frac{3}{4})^n r_n|$$

$$= |3^m + \sum_{n=0}^{m-1} (\frac{3}{4})^n r_n|$$

$$\geq 3^m - \sum_{n=0}^{m-1} 3^n$$

$$= \frac{1}{2}(3^m + 1)$$

As $m \rightarrow \infty$, $\delta_m \rightarrow 0$, but $|\frac{f(x + \delta_m) - f(x)}{\delta_m}| \rightarrow \infty$.

Hence, f is not differentiable at x . (x can be taken on all $\mathbb{R}$.)

Let X be a metric space, $x_0 \in X$, $f(x) = d(x, x_0)$ ($f: X \rightarrow \mathbb{R}$). Then f is uniformly continuous on X .

p.f: $\forall \varepsilon > 0$, $\exists \delta < \varepsilon$ s.t. $|f(x) - f(y)| = |d(x, x_0) - d(y, x_0)| \leq d(x, y) < \varepsilon$ with $d(x, y) < \delta$.

分量一致连续, 则一致连续
Theorem Let X be a metric space, A function $f = (f_1, \dots, f_d): X \rightarrow \mathbb{R}^d$ is (uniformly) continuous precisely if all the f_i ($i = 1, \dots, d$) are (uniformly) continuous.

分量 u.c. $\rightarrow$ 函数 u.c.

Proposition Let $\{f_n\}$ be a sequence of continuous functions from one metric space (X, d_1) to another (Y, d_2) , and suppose this sequence converges uniformly to another function $f: X \rightarrow Y$. Let $\{x_n\}$ be a sequence of points in X which converge to some limit x_0 . Then $f_n(x_n)$ converges to $f(x_0)$. *函数点列收敛*

p.f: Because $\{f_n\}$ converges uniformly to f , $\leftarrow$ Let $\varepsilon > 0$. *收敛*

$\exists N_1 \in \mathbb{N}$ s.t. $d_2(f_n(x), f(x)) < \frac{\varepsilon}{2}$ for $\forall n \geq N_1, x \in X$.

Because $\{f_n\}_{n \in \mathbb{N}}$ are continuous, then f is continuous.

$\exists \delta > 0$ s.t. $d_2(f(x), f(x_0)) < \frac{\varepsilon}{2}$ for any $x \in B(x_0, \delta)$.

For this δ , because $x_n \rightarrow x_0$, $\exists N_2 \in \mathbb{N}$ s.t. $d_1(x_n, x_0) < \delta$, $\forall n \geq N_2$.

Let $N = \max\{N_1, N_2\}$, for $n \geq N$, $d_2(f_n(x_n), f(x_0)) \leq d_2(f_n(x_n), f(x_n)) + d_2(f(x_n), f(x_0)) < \varepsilon$
一致连续 连续 + $\{x_n\}$ 收敛

Hence, $f_n(x_n)$ converges to $f(x_0)$.

Definition (Bounded functions) A function $f: X \rightarrow Y$ from one metric space (X, d_X) to another (Y, d_Y) is bounded if $f(X)$ is a bounded set, i.e., there exists a ball $B_{(Y, d_Y)}(y_0, R)$ in Y s.t. $f(x) \in B_{(Y, d_Y)}(y_0, R)$ for all $x \in X$.

任何有界的数列都有一个收敛子序列 $\rightarrow$ 考虑函数列

Proposition (Uniform limits preserve boundedness) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y) , and suppose that this sequence converges uniformly to another function $f: X \rightarrow Y$. If the functions $(f_n)_{n=1}^{\infty}$ are bounded on X for each n , then the limiting function f is also bounded on X .

pf: Let $\varepsilon = 1$. Because $(f_n)_{n=1}^{\infty}$ converge uniformly

$\exists N \in \mathbb{N}$ s.t. $d(f_n(x), f(x)) < 1$ for $n \geq N$

Because $(f_n(x))_{n=1}^{\infty}$ are bounded, $d(f_n(x), y_0) \leq d(f_n(x), f_n(x)) + d(f_n(x), y_0)$

Then $d(f(x), y_0) \leq d(f(x), f_n(x)) + d(f_n(x), y_0)$

Hence, $f(x)$ bounded.

任何有界函数列必有一个收敛子数列.

pf: Suppose $\{x_n\}$ is bounded by $[a, b]$.

Take $c = \frac{a+b}{2}$, then $[a, c]$ or $[c, b]$ contains infinite term of $\{x_n\}$, written $[a_1, b_1]$.

Arbitrarily taken $y_1 \in [a_1, b_1]$ from $\{x_n\}$.

We get $[a_k, b_k]_{k=1}^{\infty}$ $b_k - a_k = \frac{b-a}{2^k} \rightarrow 0$.

and $\{y_k\}_{k=1}^{\infty}$ which is a subsequence of $\{x_n\}$

Definition Let $\{f_n\}$ be a sequence of functions defined on a set E .

We say that $\{f_n\}$ is **pointwise bounded** on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, that is, if there exists a finite-valued function ϕ defined on E s.t. $|f_n(x)| < \phi(x)$ ($x \in E, n=1, 2, 3, \dots$).

We say that $\{f_n\}$ is **uniformly bounded** on E if there exists a number M s.t. $|f_n(x)| < M$ ($x \in E, n=1, 2, 3, \dots$).

Remark: ① $f_n(x)$ 逐点有界 $\rightarrow$ 存在一子列逐点收敛.

② $f_n(x)$ 一致有界 $\rightarrow$ 存在一子列逐点收敛 / 一致收敛

eg of ②: Let $f_n(x) = \sin nx$ ($0 \leq x \leq 2\pi, n=1, 2, 3, \dots$)

Suppose $\exists$ a sequence $\{n_k\}$ s.t. $\{\sin n_k x\}$ converges.

$\rightarrow \lim_{k \rightarrow \infty} (\sin n_k x - \sin n_{k+1} x) = 0$ ($0 \leq x \leq 2\pi$)

$\rightarrow \lim_{k \rightarrow \infty} (\sin n_k x - \sin n_{k+1} x)^2 = 0$ ($0 \leq x \leq 2\pi$). 积分与逐点收敛并不总可以交换

By Lebesgue's theorem (Assume $0 \leq g_n, g_n \leq f, f \in \mathbb{R}$ on $[a, b]$). ②

$g_n(x) \rightarrow g(x), \forall x \in [a, b] \rightarrow g \in \mathbb{R}$ on $[a, b]$ and $\int_a^b g(x) dx = \lim_{n \rightarrow \infty} \int_a^b g_n(x) dx$.

$\int_0^{2\pi} \lim_{k \rightarrow \infty} (\sin n_k x - \sin n_{k+1} x)^2 = \lim_{k \rightarrow \infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2$

$$2 \sin^2 x = 1 - \cos 2x$$

$$\begin{aligned} \sin^2 n_k x + \sin^2 n_{k+1} x &= 2 \sin n_k x \cdot \sin n_{k+1} x \\ &= \frac{1}{2} - \frac{1}{2} \cos 2n_k x + \frac{1}{2} - \frac{1}{2} \cos 2n_{k+1} x \\ &= 1 - \frac{1}{2} \cos 2n_k x - \frac{1}{2} \cos 2n_{k+1} x \end{aligned}$$

$$-2 \sin n_k x \cdot \sin n_{k+1} x = \cos(n_k + n_{k+1})x - \cos(n_k - n_{k+1})x$$

$$\text{But } \int_0^{2\pi} \lim_{k \rightarrow \infty} (\sin n_k x - \sin n_{k+1} x)^2 dx = 0, \quad \lim_{k \rightarrow \infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 dx = 2\pi$$

$$\begin{aligned} (\sin^2 n_k x + \sin^2 n_{k+1} x - 2 \sin n_k x \cdot \sin n_{k+1} x) &= \frac{1}{2} - \frac{1}{2} \cos 2n_k x + \frac{1}{2} - \frac{1}{2} \cos 2n_{k+1} x \\ &\quad + \cos(n_k + n_{k+1})x - \cos(n_k - n_{k+1})x \end{aligned}$$

Contradicts. Hence, there is no subsequence $\{n_k\}$ s.t. $\{\sin n_k x\}$ converges.

7.22

Definition A family $\mathcal{F}$ of complex functions f defined on a set E in a metric space X is said to be equicontinuous on E if for every $\varepsilon > 0$ there exists a $\delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$, $x \in E$, $y \in E$, and $f \in \mathcal{F}$. Here d denotes the metric of X .

Remark: $\forall f \in \mathcal{F}$ is uniformly continuous if $\mathcal{F}$ is equicontinuous.

7.24

Theorem If K is a compact metric space, if $f_n \in \mathcal{B}(K)$ for $n=1, 2, 3, \dots$, and if $\{f_n\}$ converges uniformly on K , then $\{f_n\}$ is equicontinuous on K .

Lemma: $\mathcal{B}(K)$ is the set of all complex-valued, continuous, bounded functions with domain K .

pf: Let $\varepsilon > 0$. Because $\{f_n\}$ converges uniformly,

$$\exists N \in \mathbb{N}^* \text{ s.t. } |f_n(x) - f_N(x)| < \varepsilon \text{ for } n > N.$$

Because continuous functions are uniformly continuous on compact sets,

$$\exists \delta > 0 \text{ s.t. } |f_i(x) - f_i(y)| < \varepsilon \text{ for } 1 \leq i \leq N \text{ and } d(x, y) < \delta. \quad \textcircled{*}$$

For $n > N$ and $d(x, y) < \delta$,

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < 3\varepsilon.$$

For $n \leq N$, $\textcircled{*}$

Hence, $\{f_n\}_{n=1}^{\infty}$ is equicontinuous on K .

7.23

逐点有界 $\rightarrow$ 逐点收敛子列

Theorem If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E , then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ s.t. $\{f_{n_k}(x)\}$ converges for every $x \in E$.

pf. Let $\{x_i\}, i=1, 2, 3, \dots$ be the points of E , arranged in a sequence. 逐点收敛

Because $\{f_n(x_1)\}$ is bounded, $\exists$ a subsequence $\{f_{1,k}\}$ s.t. $\{f_{1,k}(x_1)\}$ converges as $k \rightarrow \infty$. 在 x_1 处

Then through satisfying these two properties:

(a) S_n is a subsequence of S_{n-1} , for $n=2, 3, 4, \dots$

(b) $\{f_{n,k}(x_n)\}$ converges, as $k \rightarrow +\infty$.

We have $S_1: f_{1,1} \ f_{1,2} \ f_{1,3} \ f_{1,4} \ \dots$ 在 x_1 收敛

$S_2: f_{2,1} \ f_{2,2} \ f_{2,3} \ f_{2,4} \ \dots$ 从在 x_1 收敛的函数列中找出在 x_2 收敛

$S_3: f_{3,1} \ f_{3,2} \ f_{3,3} \ f_{3,4} \ \dots$ 的函数子列

$\dots$

Then $S: f_{1,1} \ f_{2,2} \ f_{3,3} \ f_{4,4} \ \dots \{f_{n,n}(x_i)\}$ converges as $n \rightarrow \infty$ for every $x_i \in E$

continuous, bounded

7.23

Theorem If K is compact, if $f_n \in \mathcal{C}(K)$ for $n=1, 2, 3, \dots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K , then 在 K 中

(a) $\{f_n\}$ is uniformly bounded on K , 可数稠密子集

(b) $\{f_n\}$ contains a uniformly convergent subsequence. 可列紧性

pf: (a) $\forall \varepsilon > 0$. Because $\{f_n\}$ is equicontinuous, $\exists \delta > 0$, s.t. $|f_n(x) - f_n(y)| < \varepsilon$ for all n , $d(x, y) < \delta$, then

Let E be a countable dense subset of K , then $\bigcup_{x \in E} B(x, \delta) = K$.

Because K is compact, $\exists p_1, \dots, p_r \in E$ s.t. $\forall x \in K \exists$ at least one p_i with $d(x, p_i) < \delta$. 有限个

Because $\{f_n\}$ is pointwise bounded, $\exists M_i < \infty$ s.t. $|f_n(p_i)| < M_i$ for all n .

Let $M = \max(M_1, \dots, M_r)$, then $|f_n(x)| \leq |f_n(x) - f_n(p_i)| + |f_n(p_i)| < \varepsilon + M$ for every $x \in K$. i.e. $\{f_n\}$ is uniformly bounded on K .

(b) By theorem 7.23 we know that $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ s.t. $\{f_{n_i}(x)\}$ converges for every $x \in E$. 有逐点收敛

Put $g_i = f_{n_i}$. We shall prove that $\{g_i\}$ converges uniformly on K .

$\forall \varepsilon > 0$. Let $V(x, \delta)$ be the set of all $y \in K$ with $d(x, y) < \delta$.

Because E is dense in K , and K is compact.

Then $\exists x_1, \dots, x_m$ in E s.t. $K \subset V(x_1, \delta) \cup \dots \cup V(x_m, \delta)$ (*)

逐点收敛 Because $\{g_i(x)\}$ converges pointwise, $\exists N$ s.t. $|g_i(x_s) - g_j(x_s)| < \varepsilon$, $i, j \geq N$, $1 \leq s \leq m$.

等度连续 By (*), if $x \in K$, then $x \in V(x_s, \delta)$ for some s , s.t. $|g_i(x) - g_j(x)| < \varepsilon$ for every i, j .

For $i, j \geq N$, $|g_i(x) - g_j(x)| \leq |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)|$
 $< 3\varepsilon$. For all $x \in K$. i.e. $g_i(x)$ convergent uniformly. 连续 收敛 连续

7.26

Theorem (The Stone-Weierstrass Theorem) If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n s.t.

$$\lim_{n \rightarrow \infty} P_n(x) = f(x) \text{ uniformly on } [a, b]. \text{ 用多项式逼近}$$

If f is real, the P_n may be taken real.

pf: WLOG. Assume $[a, b] = [0, 1]$, $f(0) = f(1) = 0$.

一种想法 Consider $g(x) = f(x) - f(0) - x[f(1) - f(0)]$ ($0 \leq x \leq 1$), then $g(0) = g(1) = 0$.

是用其它可被逼近函数 It suffices to show g is the limit of a uniformly convergent sequence of polynomials (because $f - g$ is a polynomial).

Then, define $f(x) \equiv 0$, $x \notin [0, 1]$. Then f is uniformly continuous on $\mathbb{R}$.

Let $Q_n(x) = C_n(1-x^2)^n$, $n=1, 2, 3, \dots$. C_n is chosen s.t. $\int_0^1 Q_n(x) dx = 1$.

Because $\int_0^1 (1-x^2)^n dx = 2 \int_0^1 (1-x^2)^n dx \geq 2 \int_0^{\delta} (1-x^2)^n dx \geq 2 \int_0^{\delta} (1-nx^2) dx = \frac{4}{3n} > \frac{1}{n}$

$$\Rightarrow C_n \leq \frac{1}{n}$$

For any $0 < \delta \leq |x|$, $Q_n(x) \leq \frac{1}{n} (1-\delta^2)^n$

与 x 无关, 且 n 为指数.

$$\begin{aligned} f(x) &= (1-x^2)^n - 1 + nx^2 \\ f(0) &= 1 - 1 = 0 \\ f'(x) &= n(1-x^2)^{n-1}(-2x) + 2nx \\ &= 2nx(1-x^2)^{n-1} \end{aligned}$$

$x \in (0, 1)$, $f'(x) > 0$

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$$x-1 \quad x+1$$

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Hence, $Q_n \rightarrow 0$ uniformly in $\delta \leq |x| \leq 1$.

Let $P_n(x) = \int_{-1}^1 f(x+t) Q_n(t) dt$ ($0 \leq x \leq 1$)

$P_n(x) = \int_{-x}^{1-x} f(x+t) Q_n(t) dt = \int_0^1 f(t) Q_n(t-x) dt$, which is a polynomial in x .

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|y-x| < \delta$ then $|f(y) - f(x)| < \frac{\varepsilon}{2}$

Let $M = \sup |f(x)|$, because $Q_n(x) \geq 0$, $\int_{-1}^1 Q_n(x) dx = 1$, $Q_n(x) \leq \sqrt{n}(1-\delta^2)^n$.

$$|P_n(x) - f(x)| = \left| \int_{-1}^1 f(x+t) Q_n(t) dt - \int_{-1}^1 Q_n(t) dt \cdot f(x) \right|$$

$$= \left| \int_{-1}^1 f(x+t) Q_n(t) dt - \int_{-1}^1 f(x) Q_n(t) dt \right|$$

$$\leq \int_{-1}^1 |f(x+t) - f(x)| Q_n(t) dt$$

$$\leq 2M \int_{-1}^{-\delta} Q_n(t) dt + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t) dt + 2M \int_{\delta}^1 Q_n(t) dt$$

$$\leq 2M \int_{-1}^{-\delta} \sqrt{n}(1-\delta^2)^n dt + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t) dt + 2M \int_{\delta}^1 \sqrt{n}(1-\delta^2)^n dt$$

$$= 4M \sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2}$$

$< \varepsilon$ for all large enough n .

9.2

Theorem Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then $\dim X \leq r$.

9.3

Theorem Suppose X is a vector space, and $\dim X = n$.

(a) A set E of n vectors in X spans X if and only if E is independent.

(b) X has a basis, and every basis consists of n vectors.

(c) If $1 \leq r \leq n$ and $\{y_1, \dots, y_r\}$ is an independent set in X , then X has a basis containing $\{y_1, \dots, y_r\}$.

$$|A\vec{x}| \leq \sup_{|\vec{x}|=1} |A\vec{x}| \cdot |\vec{x}|$$

Functions of Several Variables 多元函数

9.4

Definitions (Linear transformation) A mapping A of a vector space X into a vector space Y is said to be a linear transformation if

$$A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2, \quad A(c\vec{x}) = cA\vec{x}$$

for all $\vec{x}, \vec{x}_1, \vec{x}_2 \in X$ and all scalars c . We often write $A\vec{x}$ instead of $A(\vec{x})$ if A is linear.

注: ① linear transformations of X into X are called linear operators on X .

② If A is a linear operator on X which (i) is one-to-one; (ii) maps X onto X , we say A is invertible. $\exists A^{-1}$ on X , s.t. $A^{-1}(A\vec{x}) = \vec{x}$ for all $\vec{x} \in X$.

9.5

Theorem A linear operator A on a finite-dimensional vector space X is one-to-one iff the range of A is all of X .

9.6

Definitions

(a) Let $L(X, Y)$ be the set of all linear transformations of the vector space X into space Y . $L(X, X)$ simply write as $L(X)$. If $A_1, A_2 \in L(X, Y)$ and if c_1, c_2 are scalars, define $c_1A_1 + c_2A_2$ by $(c_1A_1 + c_2A_2)\vec{x} = c_1A_1\vec{x} + c_2A_2\vec{x}$ ($\vec{x} \in X$).

Then $c_1A_1 + c_2A_2 \in L(X, Y)$.

(b) If X, Y, Z are vector spaces, and if $A \in L(X, Y)$ and $B \in L(Y, Z)$, we define their product BA to be the composition of A and B :

$$(BA)\vec{x} = B(A\vec{x}) \quad (\vec{x} \in X)$$

Then $BA \in L(X, Z)$.

(c) For $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, define the norm $\|A\|$ of A to be the sup of all numbers $|A\vec{x}|$, where $\vec{x}$ ranges over all vectors in $\mathbb{R}^n$ with $|\vec{x}| \leq 1$.

$$\|A\| = \sup_{|\vec{x}| \leq 1} |A\vec{x}| \quad (\|A\| = \inf\{c > 0 : |Ax| \leq c|x|\} = \sup_{\vec{x} \in X} \frac{|Ax|}{|\vec{x}|} = \sup_{\vec{x} \in B(0,1)} |Ax|)$$

Remark: ① $|A\vec{x}| \leq \|A\||\vec{x}|$, for all $\vec{x} \in \mathbb{R}^n$. 不一定要 $|\vec{x}| \leq 1$.

② If $|A\vec{x}| \leq \lambda|\vec{x}|$ for all $\vec{x} \in \mathbb{R}^n$, then $\|A\| \leq \lambda$.

对于单个的线性算子只有 $\| \cdot \|$ 形式, 作用于向量后有 $| \cdot |$ 形式

9.7

Theorem

(a) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\|A\| < \infty$ and A is uniformly continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$.

A metric space?

(b) If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and C is a scalar, then $\|A+B\| \leq \|A\| + \|B\|$,

$|C| \|A\| = \|CA\|$. The distance between A and B defined as $\|A-B\|$.

(c) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$, then $\|BA\| \leq \|B\| \|A\|$.

Pf: (a) Let $\{\vec{e}_1, \dots, \vec{e}_n\}$ be the standard basis in $\mathbb{R}^n$, and $\vec{x} = \sum C_i \vec{e}_i$.

Suppose $|\vec{x}| \leq 1$, then $|C_i| \leq 1$ for $i = 1, \dots, n$. 有限个

Then $|A\vec{x}| = |\sum C_i A\vec{e}_i| \leq \sum |C_i| |A\vec{e}_i| \leq \sum |A\vec{e}_i| < \infty$

$\rightarrow \sup_{|\vec{x}| \leq 1} |A\vec{x}| \leq \sup_{|\vec{x}| \leq 1} \sum_{i=1}^n |A\vec{e}_i|$

$\rightarrow \|A\| \leq \sum_{i=1}^n |A\vec{e}_i| < \infty$

Because $|A\vec{x} - A\vec{y}| = |A(\vec{x} - \vec{y})| \leq \|A\| |\vec{x} - \vec{y}|$ for $\vec{x}, \vec{y} \in \mathbb{R}^n$, $\|A\| < \infty$.

只要距离 δ , 都可以用同一个 ε 来估
 A is L.C. $\rightarrow A$ is uniformly continuous.

(b) $|(A+B)\vec{x}| = |A\vec{x} + B\vec{x}| \leq |A\vec{x}| + |B\vec{x}| \leq (\|A\| + \|B\|) |\vec{x}|$

By remark ②.

$\rightarrow \|A+B\| \leq \|A\| + \|B\| \rightarrow \|A-C\| = \|(A-B) + (B-C)\| \leq \|A-B\| + \|B-C\|$

$\|CA\| = \sup_{|\vec{x}| \leq 1} |CA\vec{x}| = |C| \sup_{|\vec{x}| \leq 1} |A\vec{x}| = |C| \|A\|$.

(c) $|(BA)\vec{x}| = |B(A\vec{x})| \leq \|B\| |A\vec{x}| \leq \|B\| \|A\| |\vec{x}|$

By remark ②.

$\rightarrow \|BA\| \leq \|B\| \|A\|$

9.8

Theorem Let Ω be the set of all invertible linear operators on $\mathbb{R}^n$.

(a) If $A \in \Omega$, $B \in L(\mathbb{R}^n)$, and $\|B-A\| \|A^{-1}\| < 1$, then $B \in \Omega$.

(b) Ω is an open subset of $L(\mathbb{R}^n)$, and the mapping $A \rightarrow A^{-1}$ is continuous on Ω .

pf: (a) Put $\|A^{-1}\| = \frac{1}{\alpha}$, $\|B-A\| = \beta$. Then $\beta < \alpha$.

For $\forall \vec{x} \in \mathbb{R}^n$, $\alpha |\vec{x}| = \alpha |A^{-1} A \vec{x}| \leq \alpha \|A^{-1}\| \|A \vec{x}\| = |\vec{x}| \leq |(A-B)\vec{x}| + |B\vec{x}|$

$\leq \beta |\vec{x}| + |B\vec{x}|$

Lemma $A \in L(X, Y)$ and $Ax = 0$ only when $x = 0$, then A is 1-1 (injective).

$$\rightarrow (\alpha - \beta) \|\vec{x}\| \leq \|B\vec{x}\| \quad (\vec{x} \in \mathbb{R}^n)$$

Because $\alpha - \beta > 0$, then $B\vec{x} \neq 0$ if $\vec{x} \neq 0$.

Hence B is one-to-one.

By Theorem 9.5, $B \in \Omega$.

Because this holds for all B with $\|B - A\| < \alpha$, then we have the fact that Ω is open.

(b) Let $\vec{x} = B^{-1}\vec{y}$, then $(\alpha - \beta) \|B^{-1}\vec{y}\| \leq \|B B^{-1}\vec{y}\| = \|\vec{y}\| \quad (\vec{y} \in \mathbb{R}^n)$

$$\rightarrow \|B^{-1}\vec{y}\| \leq (\alpha - \beta)^{-1} \|\vec{y}\| \xrightarrow{\text{by remark 9}} \|B^{-1}\| \leq (\alpha - \beta)^{-1}.$$

$$\text{Because } B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$$

$$B^{-1}(I - BA^{-1}) \quad B^{-1} - A^{-1}$$

$$\text{By Theorem 9.7(c), } \|B^{-1} - A^{-1}\| \leq \|B^{-1}\| \|A - B\| \|A^{-1}\| \leq \frac{\beta}{\alpha(\alpha - \beta)}$$

Then as $\|B - A\| = \beta \rightarrow 0$, $\|B^{-1} - A^{-1}\| \rightarrow 0$. — 高斯形式

9.11

Definition Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$, and $\vec{x} \in E$.

If there $\exists$ a linear transformation A of $\mathbb{R}^n$ into $\mathbb{R}^m$ st.

$$\lim_{h \rightarrow 0} \frac{\|f(\vec{x} + h) - f(\vec{x}) - Ah\|}{\|h\|} = 0$$

linear operator

then we say that f is differentiable at $\vec{x}$, and we write $f'(\vec{x}) = A$.

If f is differentiable at every $\vec{x} \in E$, we say that f is differentiable in E .

9.12

Theorem If A_1, A_2 satisfy $\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0$, then $A_1 = A_2$

pf: Let $A = A_1 - A_2$

$$\forall h \neq 0, h \in \mathbb{R}^n, \|Ah\| = \|A_1 h - A_2 h\|$$

$$= \|h\| \frac{\|f(x+h) - f(x) - A_1 h - [f(x+h) - f(x) - A_2 h]\|}{\|h\|}$$

$$\leq \|h\| \left(\frac{\|f(x+h) - f(x) - A_1 h\|}{\|h\|} + \frac{\|f(x+h) - f(x) - A_2 h\|}{\|h\|} \right)$$

$$\rightarrow \lim_{h \rightarrow 0} \frac{\|Ah\|}{\|h\|} = 0 \quad \rightarrow \text{independent of } t(?)$$

for every $h \in \mathbb{R}^n$.

$$\text{For fixed } h \neq 0, \lim_{t \rightarrow 0} \frac{\|A(th)\|}{\|th\|} \rightarrow 0 \rightarrow \frac{\|Ah\|}{\|h\|} = 0 \rightarrow Ah = 0 \rightarrow A = 0$$

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9.15

Theorem Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$, f is differentiable at $x_0 \in E$, g maps an open set containing $f(E)$ into $\mathbb{R}^k$, and g is differentiable at $f(x_0)$. Then the mapping F of E into $\mathbb{R}^k$ defined by

$$F(x) = g(f(x)), \text{ which is differentiable at } x_0.$$

$$F'(x_0) = g'(f(x_0)) f'(x_0)$$

-Bk

pf: Put $y_0 = f(x_0)$, $A = f'(x_0)$, $B = g'(y_0)$, $u(h) = f(x_0+h) - f(x_0) - Ah$, $v(k) = g(y_0+k) - g(y_0)$, $h \in \mathbb{R}^n$, $k \in \mathbb{R}^m$.

$$\frac{|u(h)|}{|h|} = \varepsilon(h), \lim_{h \rightarrow 0} \varepsilon(h) = 0; \frac{|v(k)|}{|k|} = \eta(k), \lim_{k \rightarrow 0} \eta(k) = 0. \quad (*)$$

$$\text{Let } k = f(x_0+h) - f(x_0), \text{ then } |k| = |Ah + u(h)| \leq [\|A\| + \varepsilon(h)]|h|. \quad (**)$$

$$F(x_0+h) - F(x_0) - BAh = g(y_0+k) - g(y_0) - BAh = B(k - Ah) + v(k) = Bu(h) + v(k)$$

$$\begin{aligned} \text{By } (*), (**), \text{ for } h \neq 0, \frac{|F(x_0+h) - F(x_0) - BAh|}{|h|} &= \frac{|Bu(h) + v(k)|}{|h|} \\ &= \frac{|B\varepsilon(h)| |h| + |k| \eta(k)}{|h|} \\ &\leq |B| \varepsilon(h) + [\|A\| + \varepsilon(h)] \eta(k) \\ &\leq \|B\| \varepsilon(h) + [\|A\| + \varepsilon(h)] \eta(k). \end{aligned}$$

Let $h \rightarrow 0$, $k \rightarrow 0$, then we get $F'(x_0) = BA$.

偏导数

Definition Let Ω be open in $\mathbb{R}^d$, $f: \Omega \rightarrow \mathbb{R}$ a function. The i th partial derivative of f , in case it exists, is $D_i f(x) = \lim_{h \rightarrow 0} \frac{f(x+he_i) - f(x)}{h} = \frac{\partial f(x)}{\partial x_i}$

Definition Let Ω be open in $\mathbb{R}^d$, $f: \Omega \rightarrow \mathbb{R}$ a function, $v \in \mathbb{R}^d$, $\|v\|_2 = 1$. The directional derivative of f in the direction v is given by

$$D_v f(x) = \frac{d}{dh} f(x+tv) \Big|_{h=0} = \lim_{h \rightarrow 0} \frac{f(x+tv) - f(x)}{h} \text{ provided this limit exists.}$$

Vector field (向量场): Let Ω be an open subset in $\mathbb{R}^d$. A function $f: \Omega \rightarrow \mathbb{R}^d$ is also called a vector field, as to any point $x \in \Omega \subset \mathbb{R}^d$ a vector $f(x) \in \mathbb{R}^d$ is assigned by f .
每一点都对应一个向量

Campus

Definition Let $f: \Omega \rightarrow \mathbb{R}$ (Ω open in $\mathbb{R}^n$) be partially differentiable at $x \in \mathbb{R}^n$.

The gradient of f at x is the vector field

$$\nabla f(x) = \text{grad } f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right).$$

Rm: If $v \in \mathbb{R}^n$, $|v|=1$, $f: \Omega \rightarrow \mathbb{R}$, $f'(x)$ exists. $v = \sum_{i=1}^n v_i e_i$, $f'(x)(v) = \sum_{i=1}^n v_i f'(x)(e_i)$
 $= \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}(x) = \langle v, \nabla f(x) \rangle$

如何理解偏导数?

$f = (f_1, \dots, f_m)$, then $\frac{\partial f}{\partial x_j}(x_0) = \left(\frac{\partial f_1}{\partial x_j}(x_0), \dots, \frac{\partial f_m}{\partial x_j}(x_0) \right)$ 的一组基, $\{u_1, \dots, u_m\}$ 为 $\mathbb{R}^m$ 的一组基.

To differentiate a vector-valued function one just has to differentiate each of the components separately.

$$f'(x) e_j = \frac{\partial f}{\partial x_j} = \left(\frac{\partial f_1}{\partial x_j}, \dots, \frac{\partial f_m}{\partial x_j} \right)$$

$$= \sum_{i=1}^m \frac{\partial f_i}{\partial x_j} u_i = \sum_{i=1}^m (D_j f_i) u_i$$

在 u_i 的作用下, 每个都是 $(\frac{\partial}{\partial x_j}, 0, \dots, 0)$ $(0, \frac{\partial}{\partial x_j}, \dots, 0)$

$(\mathbb{R}^n \rightarrow \mathbb{R}^m)$ 可以被这么表示 $f(x) = \sum_{i=1}^m f_i(x) u_i$ ($x \in E$), $u_i, 1 \leq i \leq m$.

$$(D_j f_i)(x) = \lim_{t \rightarrow 0} \frac{f_i(x + t e_j) - f_i(x)}{t} \quad (\text{极限存在})$$

(x_n) , 于是 $D_j f_i$ 实际上是 f_i 关于 x_j 的导数.

9.7

Theorem Suppose $f: E \text{ (open)} \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, f is differentiable at a point $x \in E$. Then the partial derivatives $(D_j f_i)(x)$ exist, and $f'(x) e_j = \sum_{i=1}^m (D_j f_i)(x) u_i$ ($1 \leq j \leq n$).

pf: Fix j .

Because f is differentiable at x , then $f(x + t e_j) - f(x) = f'(x)(t e_j) + r(t e_j)$, where

$$\frac{|r(t e_j)|}{t} \rightarrow 0 \text{ as } t \rightarrow 0.$$

$$\text{Then } \lim_{t \rightarrow 0} \frac{f(x + t e_j) - f(x)}{t} = f'(x) e_j \xrightarrow{\text{represent } f \text{ in components}} \lim_{t \rightarrow 0} \sum_{i=1}^m \frac{f_i(x + t e_j) - f_i(x)}{t} u_i = f'(x) e_j$$

$\mathbb{R}^m$'s natural basis

$$\rightarrow \sum_{i=1}^m (D_j f_i)(x) u_i = f'(x) e_j$$

Rm: ① 所有偏导都存在无法说明函数在该点连续. $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$

② 该点连续无法说明偏导存在 (单变量同理)

分两个方向求

$$\frac{\partial}{\partial x} f(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$$

$$\frac{\partial}{\partial y} f(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$$

$$xy(x^2 + y^2)^{-1}$$

$$y(x^2 + y^2)^{-1} + xy(2x)$$

$$\frac{y}{x^2 + y^2} + 2x^2 y$$

所有偏导存在且连续, x_0 可导 (连续)

Theorem A $f: E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, $x_0 \in E$. If all the partial derivatives $\frac{\partial f}{\partial x_j}$ exist on a neighborhood of x_0 and are continuous at x_0 , then f is differentiable at x_0 , and the linear map $f'(x_0): \mathbb{R}^n \rightarrow \mathbb{R}^m$ is given by

$$f'(x_0)(v_1, \dots, v_n) = \sum_{j=1}^n v_j \frac{\partial f}{\partial x_j}(x_0)$$

pf: Let $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation given as $L(v_1, \dots, v_n) = \sum_{j=1}^n v_j \frac{\partial f}{\partial x_j}(x_0)$

We have to prove $\lim_{x \rightarrow x_0} \frac{|f(x) - f(x_0) - L(x - x_0)|}{|x - x_0|} = 0$.

i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for all $x \in B(x_0, \delta) \setminus \{x_0\}$, then $\frac{|f(x) - f(x_0) - L(x - x_0)|}{|x - x_0|} \leq \varepsilon$

i.e. $|f(x) - f(x_0) - L(x - x_0)| \leq \varepsilon |x - x_0|$ for all $x \in B(x_0, \delta) \setminus \{x_0\}$.

There exists a ball $B(x_0, r)$ on which the partial derivatives of f are defined.

Because each partial derivative is continuous at x_0 , $\exists 0 < \delta_j < r$ s.t. for

every $x \in B(x_0, \delta_j)$, $|\frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_j}(x_0)| < \frac{\varepsilon}{nm}$ $1 \leq j \leq n$

Take $\delta = \min(\delta_1, \dots, \delta_n)$, then for every $x \in B(x_0, \delta)$, $|\frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_j}(x_0)| < \frac{\varepsilon}{nm}$

Let $x \in B(x_0, \delta)$, $x = x_0 + v_1 e_1 + v_2 e_2 + \dots + v_n e_n$ ($v_1, v_2, \dots, v_n$ 是标量, 合起来的 v 是向量).

Note that $|x - x_0| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$, $|v_j| \leq |x - x_0|$ for all $1 \leq j \leq n$.

$$|f(x_0 + v_1 e_1 + \dots + v_n e_n) - f(x_0) - \sum_{j=1}^n v_j \frac{\partial f}{\partial x_j}(x_0)| \leq \varepsilon |x - x_0|$$

$f = (f_1, f_2, \dots, f_m)$ written as components.

$$|v_i e_i| = v_i$$

By mean value theorem, $f_i(x_0 + v_i e_i) - f_i(x_0) = \frac{\partial f_i}{\partial x_i}(x_0 + t_i e_i) v_i$, $t_i \in [0, v_i]$

Because $t_i \leq v_i \leq |x - x_0|$, then $t_i \in B(x_0, \delta) \cap B(x_0, \delta_j)$, then

$$|\frac{\partial f_i}{\partial x_i}(x_0 + t_i e_i) - \frac{\partial f_i}{\partial x_i}(x_0)| \leq |\frac{\partial f_i}{\partial x_i}(x_0 + t_i e_i) - \frac{\partial f_i}{\partial x_i}(x_0)| < \frac{\varepsilon}{nm}$$

$$\Rightarrow |f_i(x_0 + v_i e_i) - f_i(x_0) - \frac{\partial f_i}{\partial x_i}(x_0) v_i| \leq \frac{\varepsilon |v_i|}{nm}, 1 \leq i \leq m.$$

所有 $1 \leq i \leq m$ 都相加, 注意 $|y_1| + |y_2| + \dots + |y_m| \geq |y_1 + y_2 + \dots + y_m|$.

$$|f(x_0 + v_i e_i) - f(x_0) - \frac{\partial f}{\partial x_i}(x_0) v_i| < \frac{\varepsilon |v_i|}{n} \leq \frac{\varepsilon |x - x_0|}{n} \quad \leftarrow \text{这个 } x_0 + v_i e_i \text{ 为啥能用}$$

$$\text{The same, } |f(x_0 + v_i e_i + v_j e_j) - f(x_0 + v_i e_i) - \frac{\partial f}{\partial x_j}(x_0 + v_i e_i) v_j| \leq \frac{\varepsilon |x - x_0|}{n}$$

绝对值合并累加后得到 $|f(x_0 + v_1 e_1 + \dots + v_n e_n) - f(x_0) - \sum_{j=1}^n \frac{\partial f}{\partial x_j}(x_0) v_j| \leq \varepsilon |x - x_0|$

证毕.

$$\lambda \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + a_n \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

线性算子的矩阵

$\{x_1, \dots, x_n\}$ 是 X 的一组基, $\{y_1, \dots, y_m\}$ 是 Y 的一组基, 则 $A \in L(X, Y)$, $A \in \mathbb{R}^{m \times n}$ 决定了一组数 a_{ij} s.t. $Ax_j = \sum_{i=1}^m a_{ij} y_i$ ($1 \leq j \leq n$). X 基的分量可在 A 的作用下变成 Y 基的分量的线性组合

$$Ax_1 = a_{11}y_1 + \dots + a_{m1}y_m, \dots, Ax_n = a_{1n}y_1 + \dots + a_{mn}y_m$$

$$[A] = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

If $x = \sum_{j=1}^n c_j x_j$, $Ax = \sum_{i=1}^m (\sum_{j=1}^n a_{ij} c_j) y_i$. Ax 的系数为 $\sum_{j=1}^n a_{ij} c_j$ 单位向量 x

By Schwarz inequality, $|Ax|^2 = \sum_{i=1}^m (\sum_{j=1}^n a_{ij} c_j)^2 \leq \sum_{i=1}^m (\sum_{j=1}^n a_{ij}^2 \sum_{j=1}^n c_j^2) = \sum_{i=1}^m a_{ij}^2 |x|^2 = \sum_{i=1}^m a_{ij}^2$

$$\rightarrow |Ax| \leq (\sum_{i=1}^m a_{ij}^2)^{\frac{1}{2}} \rightarrow \|A\| \leq (\sum_{i=1}^m a_{ij}^2)^{\frac{1}{2}}$$

Rm: If S is a metric space, if $a_{11}, \dots, a_{mn}$ are real continuous functions on S , and if, for each $p \in S$, A_p is the linear transformation $A_p: \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose matrix has entries $a_{ij}(p)$, then the mapping $p \mapsto A_p: S \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous.

pf: Let $p, q \in S$, $\|A_p - A_q\| \leq (\sum_{i,j} (a_{ij}(p) - a_{ij}(q))^2)^{\frac{1}{2}} \rightarrow 0$.

9.19

Theorem Suppose f maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in E , and there is a real number M s.t. $\|f'(x)\| \leq M$ for every $x \in E$.

Then $|f(b) - f(a)| \leq M|b - a|$ for all $a \in E, b \in E$.

pf: Fix $a \in E, b \in E$. Define $r(t) = (1-t)a + tb$ 定义一个曲线? for all $t \in \mathbb{R}$ s.t. $r(t) \in E$

Because E is convex, $r(t) \in E$ if $0 \leq t \leq 1$.

Let $g(t) = f(r(t))$, then $g'(t) = f'(r(t)) \cdot r'(t) = f'(r(t))(b-a)$

$|g'(t)| = |f'(r(t))(b-a)| \leq \|f'(r(t))\| |b-a| \leq M|b-a|$ for $t \in [0, 1]$

Because $|g(1) - g(0)| \leq |g'(t)|_{\max} (1-0) = M|b-a|$, $g(1) = f(b)$, $g(0) = f(a)$.

$$\rightarrow |f(b) - f(a)| \leq M|b-a|$$

9.20

Definition A differentiable mapping f of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is said to be continuously differentiable in E if f' is a continuous mapping of E into $L(\mathbb{R}^n, \mathbb{R}^m)$.

① f 可微, ② f' 连续.

$$f' \in L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m))$$

Rm: 更确切地说: $f \in \mathcal{C}'(E)$

$$\Leftrightarrow \forall x, y \in E, \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } |x - y| < \delta \text{ then } \|f'(y) - f'(x)\| < \varepsilon.$$

9.21

Theorem Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Then $f \in \mathcal{C}'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq n$.

$\Rightarrow$ Because $\forall x \in E$, $f'(x)$ exists, then $f'(x)v_i = \frac{\partial f}{\partial x_i}$ exists.

$$\begin{aligned} \left| \frac{\partial f}{\partial x_i}(x) - \frac{\partial f}{\partial x_i}(y) \right| &= |f'(x)v_i - f'(y)v_i| \\ &= |(f'(x) - f'(y))v_i| \\ &\leq \|f'(x) - f'(y)\| |v_i| \end{aligned}$$

for $\forall i$, $\|f'(x) - f'(y)\| \rightarrow 0$ as $x \rightarrow y$.

$\Leftarrow$ Because $\frac{\partial f}{\partial x_i}$ exists and continuous, then $f'(x)$ exists

$$\|f'(x) - f'(y)\| \leq \left(\sum_{i,j} \left(\frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y) \right)^2 \right)^{\frac{1}{2}} \rightarrow 0 \text{ as } x \rightarrow y.$$

矩阵的每个元素 $\text{By } \|A\| \leq \left(\sum_{i,j} a_{ij}^2 \right)^{\frac{1}{2}}$

9.22

Definition Let X be a metric space, with metric d . If φ maps X into X and if there is a number $c < 1$ s.t. $d(\varphi(x), \varphi(y)) \leq c d(x, y)$ for all $x, y \in X$, then φ is said to be a contraction of X into X . 压缩 uniformly continuous

9.23

完备度量空间上的压缩映射存在且仅存在一个不动点

Theorem If X is a complete metric space, and if φ is a contraction of X into X , then there exists one and only one $x \in X$ s.t. $\varphi(x) = x$.

If $\varphi(x) = x$, $\varphi(y) = y$, by definition, $d(\varphi(x), \varphi(y)) = d(x, y) \leq c d(x, y) \rightarrow d(x, y) = 0$.

pf: Pick $x_0 \in X$ arbitrarily, Then define $\{x_n\}$ recursively by setting $x_{n+1} = \varphi(x_n)$

$$\text{For } n \geq 1, c < 1, d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1})) \leq c d(x_n, x_{n-1})$$

Hence induction gives $d(x_{n+1}, x_0) \leq c^n d(x_1, x_0)$.

$$\leq [(1-c)^{-1} d(x_1, x_0)] c^n$$

小于一个数

$$\text{For } n < m, d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \leq (c^n + c^{n+1} + \dots + c^{m-1}) d(x_1, x_0)$$

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n, m \geq N, d(x_n, x_m) < \varepsilon$$

$$x^2 - y^2 = f(x) - f(y) = 1 - \frac{1}{4} = \frac{3}{4}$$

No.

Date

Thus $\{x_n\}$ is a Cauchy sequence.

Because X is complete, $\lim_{n \rightarrow \infty} x_n = x$, $x \in X$.

Because $d(\varphi(x), \varphi(y)) \leq c d(x, y)$, $c < 1$, then $\varphi(x)$ is continuous on X .

$$\varphi(x) = \lim_{n \rightarrow \infty} \varphi(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x$$

9.24 反函数定理

所有偏导数都连续

雅可比矩阵

Theorem Suppose f is a C^1 -mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $f'(a)$ is invertible for some $a \in E$, and $b = f(a)$. Then

(a) there exist open sets U and V in $\mathbb{R}^n$ such that $a \in U$, $b \in V$, f is one-to-one on U , and $f(U) = V$;

(b) if g is the inverse of f [by (a), f exists], defined in V by $g(f(x)) = x$ ($x \in U$),

then $g \in C^1(V)$. 反函数也是连续可微的, 且反函数的导数是雅可比矩阵的逆

(Interpretation: The system of n equations $y_i = f_i(x_1, \dots, x_n)$ can be solved for $x_1, \dots, x_n$ in terms of $y_1, \dots, y_n$, if we restrict x and y to small enough neighborhoods of a and b ;

the solutions are unique and continuously differentiable.

$$\text{一维下, } g'(b) = \frac{1}{f'(a)}$$

9.25

Theorem If f is a C^1 -mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and if $f'(x)$ is invertible for every $x \in E$, then $f(W)$ is an open subset of $\mathbb{R}^n$ for every open set $W \subset E$.

$f^{-1}(f^{-1}(W))$ 得到了开映射性质

9.27

Theorem If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and if A_x is invertible, then there corresponds to every $k \in \mathbb{R}^m$ a unique $h \in \mathbb{R}^n$ s.t. $A(h, k) = 0$.

This h can be computed from k by the formula $h = -(A_x)^{-1} A_y k$.

pf: Every $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ can be split into two linear transformations A_x, A_y , defined by $A_x h = A(h, 0)$, $A_y k = A(0, k)$ for any $h \in \mathbb{R}^n$, $k \in \mathbb{R}^m$.

Then $A_x \in L(\mathbb{R}^n, \mathbb{R}^n)$, $A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$, $A(h, k) = A_x h + A_y k$

Hence, $A(h, k) = 0$ iff $A_x h + A_y k = 0$. A_x invertible $\rightarrow h = -(A_x)^{-1} A_y k$

KOKUYO

No.

Date

可逆 $\Leftrightarrow$ 线性 $\Leftrightarrow$ 零空间只有零

9.25 隐函数定理

Theorem Let f be a $\mathcal{C}^1$ -mapping of an open set $E \subset \mathbb{R}^{n+m}$ into $\mathbb{R}^n$, such that $f(a,b)=0$ for some point $(a,b) \in E$.

Put $A = \frac{\partial f}{\partial x}(a,b)$ and assume that A is invertible. 行列式不为0.

Then there exist open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^m$, with $(a,b) \in U$ and $b \in W$, having the following property:

To every $y \in W$ corresponds a unique x st. $(x,y) \in U$ and $f(x,y)=0$. If this x is defined to be $g(y)$, then g is a $\mathcal{C}^1$ -mapping of W into $\mathbb{R}^n$, $g(b)=a$,

$f(g(y), y)=0$ ($y \in W$), $g'(b) = - (Ax)^{-1} A_y - \frac{f_y(a,b)}{f_x(a,b)}$ equations $\left\{ \begin{array}{l} f_1(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \\ \vdots \\ f_n(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \end{array} \right.$

(Interpretation: $f(x,y)=0$ can be written as a system of n equations

If these holds for $x=a, y=b$, then by the theorem (9.25) can be solved for

$x_1, \dots, x_n$ in terms of $y_1, \dots, y_m$, for every y near b , and that these solutions are continuously differentiable functions of y .)

$f(x,y)=0$ 定 x 一个隐函数 $x=g(y)$, 在 (a,b) 附近, 可将 x 看成是 y 的函数.

当 A 可逆, 且能确保局部唯一解的存在